Notations And Abbrevations مصطلحات واختصارات

$\alpha \equiv$ Alpha , $\beta \equiv$ Beta , γ or $\Gamma \equiv$ Gamma, δ or $\Delta \equiv$ delta

$\theta \equiv$ Theta , $\lambda \equiv$ lemda , $\eta \equiv$ Eata , $\xi \equiv$ Zeeta, $\mu \equiv$ Mou ,

σ or $\Sigma \equiv$ Segma , π or $\Pi \equiv$ Pi , ϕ or $\Phi \equiv$ Fi

ψ or $\Psi \equiv$ Epsi , $\epsilon \equiv$ Epsilent , $\tau \equiv$ Tow , $\rho \equiv$ Row

$\nabla \equiv$ Carol , ω or $\Omega \equiv$ Omega .

$1^{st} \equiv$ First , $2^{nd} \equiv$ second , $3^{rd} \equiv$ Third , $4^{th} \equiv$ Fourth, ...

no. $\equiv$ Number , no^{s}. $\equiv$ Numbers , +ive $\equiv$ Positive , −ive $\equiv$ negat

$\ni \equiv$ Such that , $\forall \equiv$ For each , $\exists \equiv$ There exist ,

w.r.t. $\equiv$ with respect to , Lim $\equiv$ Limit , D $\equiv$ Domain

R $\equiv$ Range , Int. $\equiv$ Intercept , Symm. $\equiv$ Symmetry or Symmetric

Asy. $\equiv$ Asymptote , V. $\equiv$ Vertical , H. $\equiv$ Horizonal ,

$\mathbb{R} \equiv$ Set of real numbers $= \{x : -\infty < x < \infty\}$

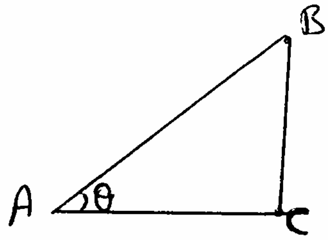
$\mathbb{C} \equiv$ Set of Complex numbers .

$=$ Equal , $\equiv$ Identical , $\geq$ Greater than or equal

$\leq$ Less than or equal , $\Rightarrow$ Implies , $\rightarrow$ Approach .

Some Trigonometric Identities

$$\sin\theta = \frac{BC}{AB} \quad , \quad \cos\theta = \frac{AC}{AB}$$



$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{BC}{AC}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{AC}{BC} .$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{AB}{AC} \quad , \quad \csc\theta = \frac{1}{\sin\theta} = \frac{AB}{BC} .$$

$$-1 \leq \sin\theta \leq 1 \quad \text{and} \quad -1 \leq \cos\theta \leq 1$$

$$-\infty \leq \tan\theta \leq \infty \quad \text{and} \quad -\infty \leq \cot\theta \leq \infty$$

$$\{ \sec\theta \leq -1 \text{ or } \sec\theta \geq 1 \} \text{ and } \{ \csc\theta \leq -1 \text{ or } \csc\theta \geq 1 \}$$

$$\sin^2\theta + \cos^2\theta = 1 \quad , \quad \sec^2\theta = \tan^2\theta + 1 \quad , \quad \csc^2\theta = \cot^2\theta + 1$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \sin\theta_2 \cos\theta_1$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}$$

$$\sin(2\theta) = 2 \sin\theta \cos\theta .$$

$$\sin^2(\theta) = \frac{1 - \cos 2\theta}{2} \quad , \quad \cos^2 = \frac{1 + \cos 2\theta}{2}$$

$$\sin(-\theta) = -\sin\theta \quad , \quad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta .$$

The solution of $ax^2+bx+c=0$ is $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

The Indeterminate Forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞, ∞^0, $\infty-\infty$, $0.\infty$

Equation of A straight Line

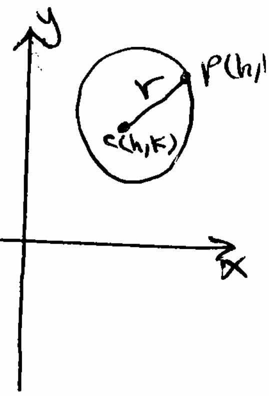
The eq. of a st. Line is $ax+by+c=0$ where a, b, c are constants.

Circle Is the locus of all points in plane whose distance from fixed point is constant.

The fixed point is called the center of the circle and denotee by $C(h,k)$ and the constant distance is called the radius of the circle and denoted by r.

The eq. of the circle with center at (h,k) and radius r is

$$r^2=(x-h)^2+(y-k)^2 \qquad (1)$$



Note If $h=k=0$, then eq.(1) becomes

$$r^2=x^2+y^2$$

Inequalities المتراجحات

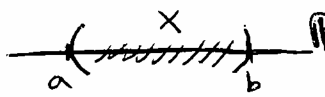
If a and b are real no.'s, then one of the following is true: $a > b$ or $a = b$ or $a < b$

Notes: (1) If $a > b$ then $-a < -b$.
(2) If $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

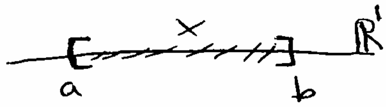
Intervals الفترات

Defn. An interval is a set of no.'s x having one of the following form:

(i) Open interval: $a < x < b \equiv (a,b)$



(ii) Closed interval: $a \leq x \leq b \equiv [a,b]$

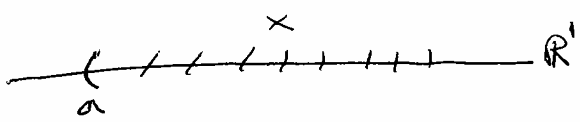


(iii) Half open from the left or half close from the right:
$a < x \leq b \equiv (a,b]$.

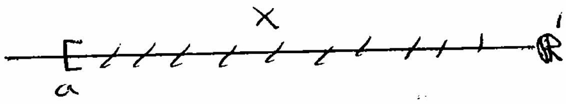
(iv) Half close from the left or half open from the right:
$a \leq x < b \equiv [a,b)$.

Notes:

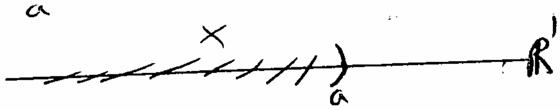
(1) $a < x < \infty \equiv a < x \equiv (a,\infty)$



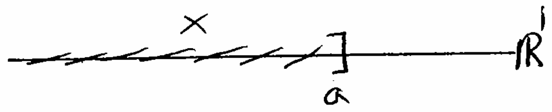
(2) $a \leq x < \infty \equiv a \leq x \equiv [a,\infty)$



(3) $\infty < x < a \equiv x < a \equiv (-\infty, a)$



(4) $\infty < x \leq a \equiv x \leq a \equiv (-\infty, a]$



Absolute Value القيمة المطلقة

Defn: The absolute value of a real no. x is define as

$$|x| = \begin{cases} x & , \quad x \geq 0 \\ -x & , \quad x < 0 \end{cases}$$

Properties of Absolute Values : خصائص القيم المطلقة

1. $|x \cdot y| = |x| \cdot |y|$ and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

2. $|-x| = |x|$.

3. $|x+y| \leq |x| + |y|$.

4. $|x| < a$ mean $-a < x < a$.

5. $|x| \leq a$ mean $-a \leq x \leq a$.

6. $|x| > a$ mean $x < -a$ or $x > a$.

7. $|x| \geq a$ mean $x \leq -a$ or $x \geq a$.

Example Find the solution set of the following ineq.^{s} :

(1) $\left|\frac{3x+1}{2}\right| < 1$, (2) $|x-1| \geq 5$.

Solu.

(1) $\left|\frac{3x+1}{2}\right| < 1 \Rightarrow -1 < \frac{3x+1}{2} < 1 \Rightarrow -2 < 3x+1 < 2$

$\Rightarrow -3 < 3x < 1 \Rightarrow -1 < x < \frac{1}{3}$

(2) $|x+1| \geq 5 \Rightarrow x-1 \leq -5$ or $x-1 \geq 5 \Rightarrow x \leq -4$

or $x \geq 6$

Graphs And functions :

Defn. : The solution set or Locus of an equation in two unknown Consists of all points in the plane whose Coordinates satisfy the eq.

A geometrical representation of the locus is called the graph of the equation.

EX. Sketch the graph of the following eq.'s :

(1) $2x+3y=6$.

(2) $y = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \end{cases}$

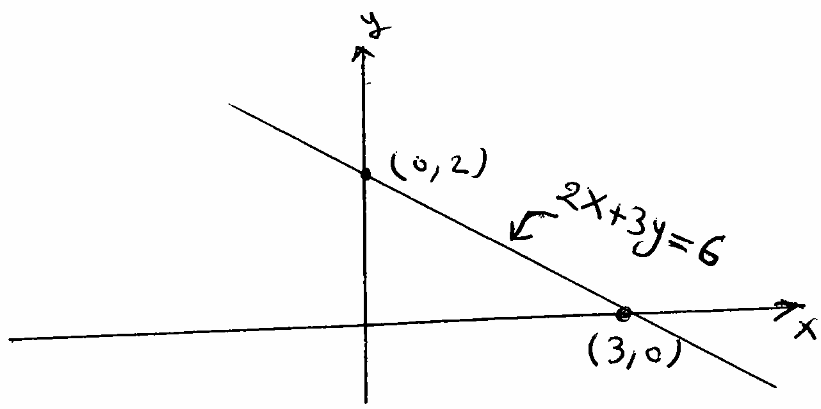
(3) $y = \begin{cases} -x & , \ x<0 \\ x^2 & , \ 0<x\leq 1 \\ 1 & , \ 1<x \end{cases}$

(4) $y = |x^2-1|$

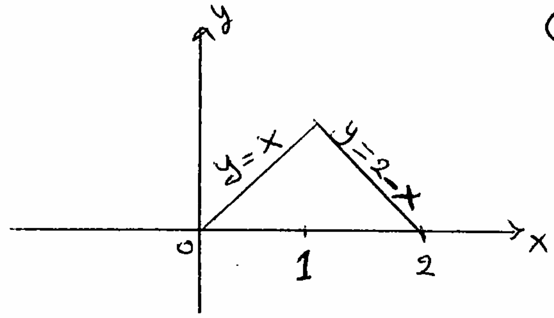
(5) $16x^2+25y^2=400$.

Solu.

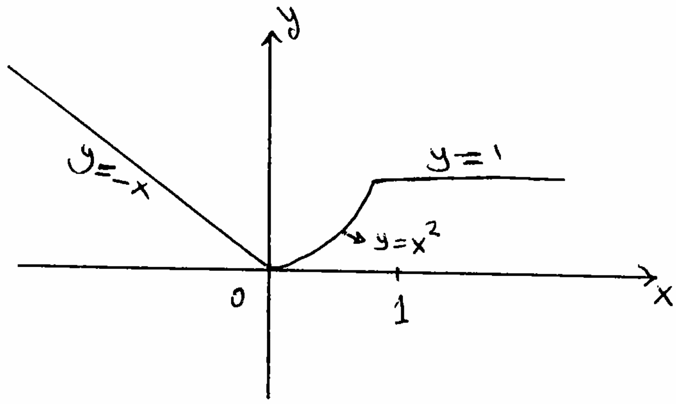
(1) $2x+3y=6$



(2) $y = \begin{cases} x & , \ 0 \le x \le 1 \\ 2-x & , \ 1 < x \le 2 \end{cases}$



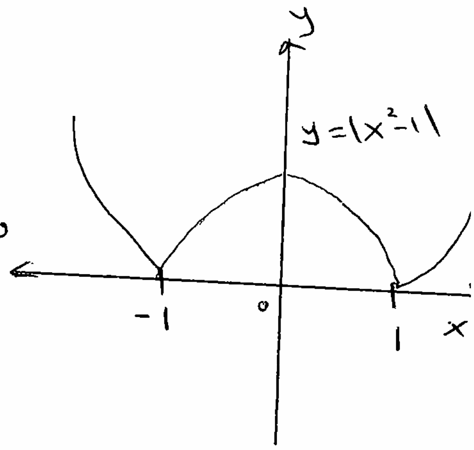
(3) $y = \begin{cases} -x & , \ x < 0 \\ x^2 & , \ 0 \le x \le 1 \\ 1 & , \ 1 < x \end{cases}$



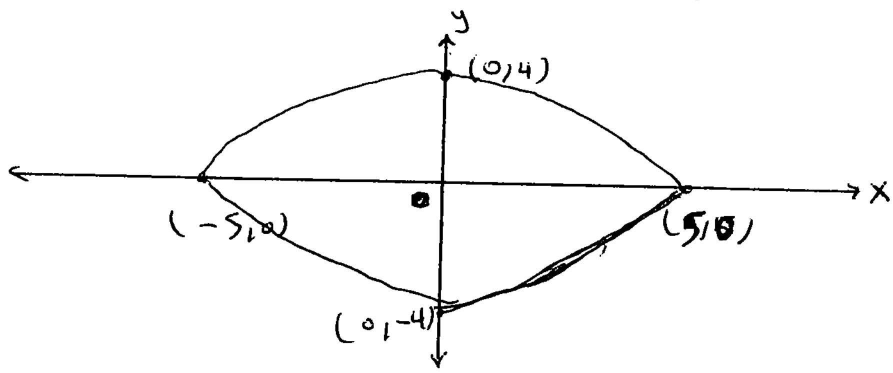
(4) $y = |x^2 - 1| = \begin{cases} x^2-1 & , \ x^2-1 \ge 0 \\ -(x^2-1) & , \ x^2-1 < 0 \end{cases}$

$= \begin{cases} x^2-1 & , \ (x-1)(x+1) \ge 0 \\ 1-x^2 & , \ (x-1)(x+1) < 0 \end{cases}$

$= \begin{cases} x^2-1 & , \ x \le -1 \ \text{or} \ x \ge 1 \\ 1-x^2 & , \ -1 < x < 1 \end{cases}$



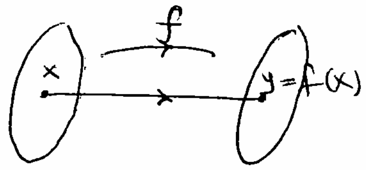
(5) $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$, ellipse



دالة

<u>Defn</u>: (Function) : A function f from a set D to a set R is a rule that assigns a single element $y \in R$ to each element $x \in D$.

<u>Note</u> The element $y \in R$ denoted by $f(x)$, the set D is called the domain of f, and the set R is called the range of f.



D R

<u>Defn.</u> $f(x)$ is an even function if $f(-x) = f(x)$.
$f(x)$ is an odd function if $f(-x) = -f(x)$.

<u>EX.</u>
(1) $f(x) = x^2 \cos x \Rightarrow f(-x) = (-x)^2 \cos(-x)$
$= x^2 \cos x = f(x)$

$\therefore$ $f(x)$ is even function.

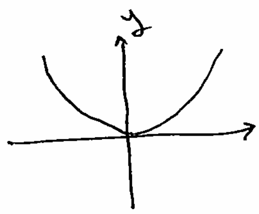
(2) $f(x) = \dfrac{x^2 - 1}{\sin x} \Rightarrow f(-x) = \dfrac{(-x)^2 - 1}{\sin(-x)} = \dfrac{x^2 - 1}{-\sin x} = -f(x)$

$\therefore$ $f(x)$ is odd function.

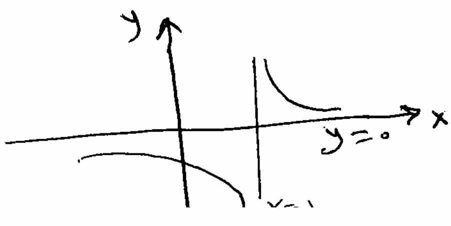
<u>Note</u> : We may define
The domain D is the set of all values of x for which y is define
The range R " " " " " " y " x " ".

<u>EX.</u> Find the domain and the range of the following functions:

(1) $y = f(x) = x^2$, D: all x , R: $y \geq 0$.

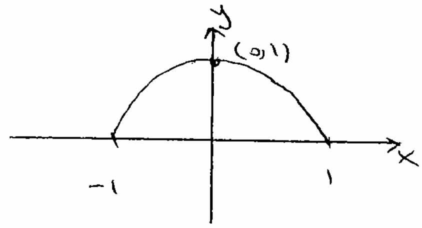


(2) $y = \dfrac{1}{x-1}$, D: $x \neq 1$



$X = \dfrac{y+1}{y}$, $R : y \neq 0$

③ $y = \sqrt{4 - x^2}$, $D : -2 \leq X \leq 2$
$R : 0 \leq y \leq 2$



④ $y = f(x) = \sqrt{x^2 - 4x + 3}$

$x^2 - 4X + 3 \geqslant 0 \implies D : X \leq 1$ or $X \geqslant 3$

$y^2 = x^2 - 4X + 3 \implies x^2 - 4x + 3 - y^2 = 0$

$X = \dfrac{4 \pm \sqrt{16 - 4(3 - y^2)}}{2} = \dfrac{4 \pm \sqrt{4 + 4y^2}}{2} = 2 \pm \sqrt{1 + y^2}$

$\therefore$ R: all y.

⑤ $y = \sqrt{2 - \sqrt{x}}$

For $\sqrt{x}$ it must be $x \geqslant 0$

$2 - \sqrt{x} \geqslant 0 \implies 2 \geqslant \sqrt{x} \implies 4 \geqslant x$

$\therefore$ $D : 0 \leq X \leq 4$

$X = (2 - y^2)^2$, R: all y.

Intercepts, Symmetry, and Asymptotes

(1) To find x-intercepts, set $y = 0$ and solve for y.
To find y-intercepts, set $x = 0$ and solve for x.

(2) The locus is symmetric w.r.t the

(i) x-axis $(x, -y) \iff (x, y)$
(ii) y-axis $(-x, y) \iff (x, y)$
(iii) origin $(-x, -y) \iff (x, y)$

(3) (i) A line $x=a$ near which a locus goes of f to ∞ is Called V. Asy.

(ii) A line $y=b$ near which a locus goes of f to ∞ is called H. Asy.

EX. Find the domain ~~and~~, the range, intercepts, symmetry, and asymptotes if they exist for the following functions. Sketch.

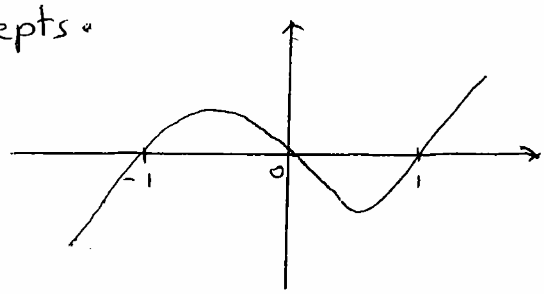
(1) $y=f(x)=x^3-x$, D: all x , R: all y

$(0,0), (1,0), (-1,0)$ are x-intercepts.

$(0,0)$ is y-intercept.

Symmetric w.r.t. origin only

No asymptotes.



(2) $y=f(x)=\dfrac{1}{x^2-1}$, D: $x \neq \pm 1$

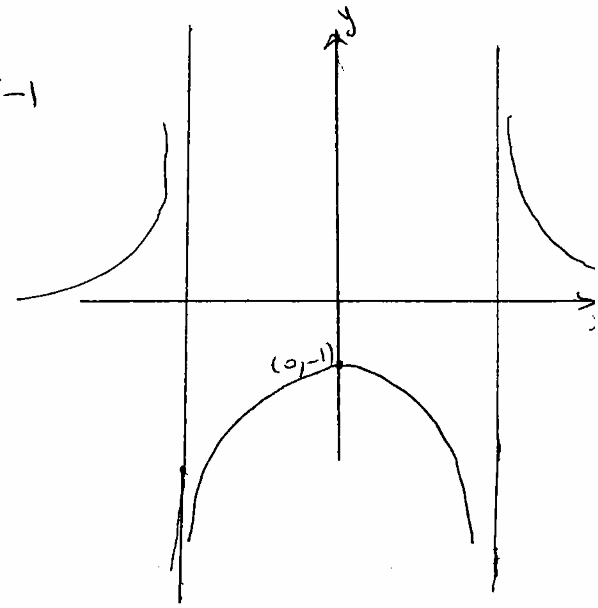
$x=\pm\sqrt{\dfrac{y+1}{y}}$, R: $y>0$ or $y \leq -1$

$(0,-1)$ is y-intercepts

Symm. w.r.t. y-axis only

$x=\pm 1$, V. Asy.

$y=0$, H. Asy.



Limit And Continuity الغاية والاستمرارية

Notation

When $f(x)$ tends to the number L as x tends to the number a we write $f(x) \longrightarrow L$ as $x \longrightarrow a$

or $\lim\limits_{x \to a} f(x) = L$.

EX.(1) let $f(x) = 2x+5$

Evaluate $f(x)$ at $x = 1.1, 1.01, 1.001, 1.0001, \ldots$

$f(1.1) = 2(1.1)+5 = 7.2$

$f(1.01) = 2(1.01)+5 = 7.02$

$f(1.001) = 2(1.001)+5 = 7.002$

$f(1.0001) = 2(1.0001)+5 = 7.0002$

$\vdots$

We see that $f(x) \longrightarrow 7$ as $x \longrightarrow 1$

or $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (2x+5) = 2(1)+5 = 7$

EX.(2) If $f(x) = \dfrac{x^2-3x+2}{x-2}$, $x \neq 2$. find $\lim\limits_{x \to 2} f(x)$.

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^2-3x+2}{x-2} = \dfrac{4-6+2}{2-2} = \dfrac{0}{0}$ meaning less

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^2-3x+2}{x-2} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}} = \lim\limits_{x \to 2} (x-1) = 2-1 = 1$.

EX.(3) Evaluate the following limits, if they exist.

1. $L = \lim\limits_{x \to -1} \dfrac{\sqrt{2+x} - 1}{x+1}$, $x \neq -1$, $x \geq -2$

$$L=\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1}\cdot\frac{\sqrt{2+x}+1}{\sqrt{2+x}+1}=\lim_{x\to -1}\frac{(2+x-1)}{(x+1)(\sqrt{2+x}+1)}$$

$$=\frac{1}{\sqrt{2-1}+1}=\frac{1}{1+1}=\frac{1}{2}.$$

2. $\lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}}$, $x\neq 2$, $x\geq 0$

$$L=\lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}}\cdot\frac{2+\sqrt{2x}}{2+\sqrt{2x}}=\lim_{x\to 2}\frac{(2-x)(2+\sqrt{2x})}{4-2x}$$

$$=\lim_{x\to 2}\frac{(2-x)(2+\sqrt{2x})}{2(2-x)}=\frac{2+\sqrt{4}}{2}=\frac{2+2}{2}=2.$$

H.W

3. $\lim_{x\to 3}\frac{\sqrt{3x}-3}{x-3}$, $x\neq 3$

H.W

4. $\lim_{x\to 2}\frac{x^4-2x^2-8}{x^2-4}$, $x\neq 2$

H.W

5. $\lim_{x\to a}\frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}$, $x\neq a$

H.W

6. $\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x-2}-\frac{1}{2}\right)$, $x\neq 0, 2$.

H.W

7. $\lim_{x\to 0}\frac{(1+x)^{3/2}-1}{x}$, $x\neq 0$.

Theorems On Limits (Calculation Technique)

1. Uniqueness of limit

If $\lim_{x\to a} f(x)=L_1$, and $\lim_{x\to a} f(x)=L_2 \Rightarrow L_1=L_2$.

2. Limit of Constant

If $f(x)=C$, C is Constant then $\lim_{x\to a} f(x)=\lim_{x\to a} C = C.$

3. Obvious limit

If $f(x)=x$ then $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$.

4. limit of Sum

If $f(x)=f_1(x)\pm f_2(x)\pm f_3(x)\pm\cdots\pm f_n(x)$ and $\lim_{x\to a} f_i(x)=L_i$,

$i=1,2,\cdots,n$ then

$$\lim_{x\to a} f(x)=\lim_{x\to a}\left[f_1(x)\pm f_2(x)\pm\cdots\pm f_n(x)\right]$$

$$=\lim_{x\to a} f_1(x)\pm\lim_{x\to a} f_2(x)\pm\cdots\pm\lim_{x\to a} f_n(x)$$

$$=L_1\pm L_2\pm\cdots\pm L_n=\sum_{i=1}^{n} L_i$$

5. Limit of a product

If $f(x)=f_1(x)\cdot f_2(x)\cdot\cdots\cdot f_n(x)$ and $\lim_{x\to a} f_i(x)=L_i$

$i=1,2,\cdots,n$ then $\lim_{x\to a} f(x)=\lim_{x\to a}[f_1(x)\cdot f_2(x)\cdot\cdots\cdot f_n(x)]$

$$=\lim_{x\to a} f_1(x)\cdot\lim_{x\to a} f_2(x)\cdot\cdots\cdot\lim_{x\to a} f_n(x)$$

$$=L_1\cdot L_2\cdot\cdots\cdot L_n=\prod_{i=1}^{n} L_i.$$

6. Limit of a Quotient

If $f(x)=\dfrac{g(x)}{h(x)}$ and $\lim_{x\to a} g(x)=L_1$, and $\lim_{x\to a} h(x)=L_2\neq 0$

then $$\lim_{x\to a} f(x)=\lim_{x\to a}\frac{g(x)}{h(x)}=\frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)}=\frac{L_1}{L_2}.$$

EX. (4) Evaluate the following limits:

(i) $\lim_{x\to 1}\dfrac{x^3-1}{x-1}$, $x\neq 1$

$$\lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}=\lim_{x\to 1}(x^2+x+1)=(1)^2+1+1=3$$

(ii) $\lim\limits_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$, $h \neq 0$

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h}\,\frac{x-x-h}{(x+h)\,x} = \lim_{h\to 0} \frac{-h}{h\,(x+h)\,x}$$

$$= -\lim_{h\to 0} \frac{1}{x(x+h)} = -\frac{1}{x(x+0)} = -\frac{1}{x^2}.$$

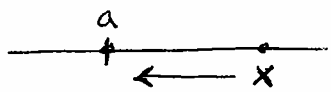
H.W

(iii) $\lim\limits_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$, $h \neq 0$.

One Sided and Two Sided limits (Right limits and Left limits)

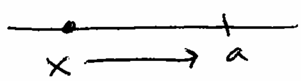
Some times the Values of a function $f(x)$ tend to different limits as x tends a from different sides. When this happens, we the limit of $f(x)$ as x approaches a from the right by the Right-hand limit and denoted by

$$\lim_{x\to a^+} f(x) = L$$



and the limit of $f(x)$ as x approaches a from the left by the left-hand limit and denoted by

$$\lim_{x\to a^-} f(x) = L$$

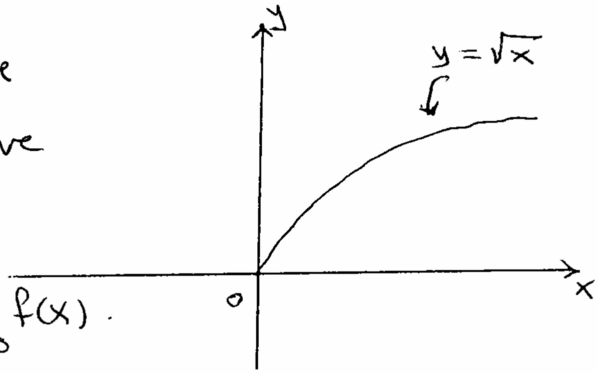


Note From uniqueness theorem of the limit, we know that if Limit exist then it is unique, so that

$\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.

EX.(5) $f(x)=\sqrt{x}$, $D: x \geq 0$. Find $\lim\limits_{x\to 0} f(x) = ?$

since $\sqrt{x}$ is not define for $-$ive value of x, so we restrict to $+$ive value of x.



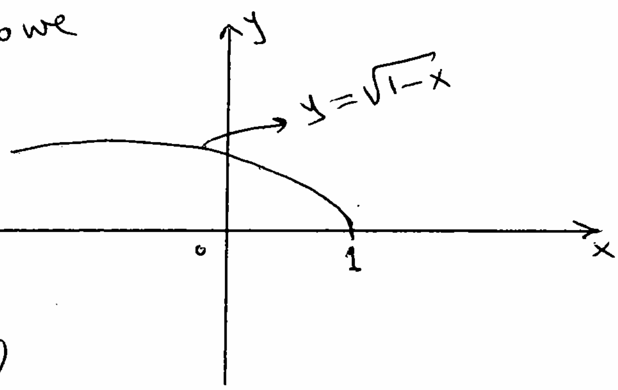
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0 = \lim_{x\to 0} f(x).$$

(This example of one-sided limit).

EX.(6) $f(x)=\sqrt{1-x}$, $D: x \leq -1$. Find $\lim\limits_{x\to 1} f(x) = ?$

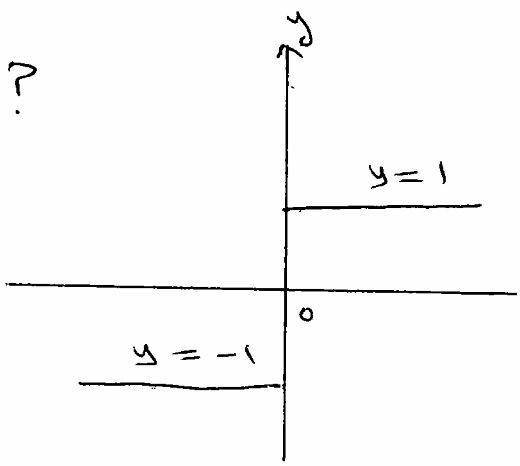
since $\sqrt{1-x}$ is not define for $x>1$, so we restrict to values of $x \leq 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = \sqrt{1-1} = \sqrt{0} = 0$$
$$= \lim_{x\to 1} f(x)$$



(This example of one-sided limit)

EX.(7) $f(x) = \dfrac{x}{|x|}$. Find $\lim\limits_{x\to 0} f(x) = ?$



since $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$\therefore \quad f(x) = \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 1 = 1$ and $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (-1) = -1$

Since $\lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$, the $\lim\limits_{x\to 0} f(x)$ does not exist.

(This example of two-sided limit)

H.W

EX. (8) $f(x) = \dfrac{x\sqrt{x^2+1}}{|x|}$, $x \neq 0$. Find $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0} f(x)$.

EX. (9) $f(x) = \dfrac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$. What is the domain. $\lim\limits_{x\to 2^-} f(x)$, $\lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2} f(x)$.

D: $-2 \leq x < 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}} = \lim_{x\to 2^-} \frac{\cancel{\sqrt{2-x}}\,\sqrt{2+x}}{\cancel{\sqrt{2-x}}\,\sqrt{3-x}}$$

$$= \lim_{x\to 2^-} \frac{\sqrt{2+x}}{\sqrt{3-x}} = \frac{\sqrt{2+2}}{\sqrt{3-2}} = \sqrt{4} = 2.$$

$\lim\limits_{x\to 2^+} f(x)$ is not define, so $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2^-} f(x) = 2$.

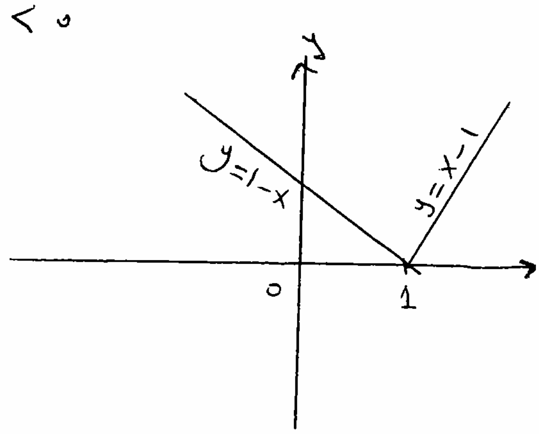
H.W

EX. (10) $f(x) = |x-1|$. Find $\lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1} f(x)$.

solu.

$$f(x) = |x-1| = \begin{cases} (x-1), & x-1 \geq 0 \\ -(x-1), & x-1 < 0 \end{cases}$$

$$= \begin{cases} x-1, & x-1 \geq 0 \\ & x \geq 1 \\ 1-x, & x-1 < 0 \\ & x < 1 \end{cases}$$



$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 1-1 = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} |x-1| = 0$$

Limits At Infinity

We note that when the limit of a function $f(x)$ exist as x approachs infinity, we write $\lim\limits_{x\to\infty} f(x)=L$.

So, we write

$\lim\limits_{x\to\infty} f(x)=L$ for +ive values of x and $\lim\limits_{x\to-\infty} f(x)=L$ for −ive values of x.

For one-sided and Two-sided limits, we have

$\lim\limits_{x\to\infty} f(x)=L$ if and only if $\lim\limits_{x\to+\infty}=L$ and $\lim\limits_{x\to-\infty} f(x)=L$

Some Obvious limits

(1) If K is constant, then $\lim\limits_{x\to+\infty} K=K$ and $\lim\limits_{x\to-\infty} K=K$.

(2) $\lim\limits_{x\to\infty}\frac{1}{x}=0$, $\lim\limits_{x\to+\infty}\frac{1}{x}=0$, and $\lim\limits_{x\to-\infty}\frac{1}{x}=0$.

(3) $\lim\limits_{x\to0}\frac{1}{x}=\infty$, $\lim\limits_{x\to0^+}\frac{1}{x}=+\infty$, and $\lim\limits_{x\to0^-}\frac{1}{x}=-\infty$.

EX.(11) Find the following limits:

(1) $$\lim_{x\to\infty}\frac{x}{2x+3}=\lim_{x\to\infty}\frac{1}{2+\frac{3}{x}}=\frac{1}{2+0}=\frac{1}{2}.$$

(2) $$\lim_{x\to\infty}\frac{2x^2+3x+5}{5x^2-4x+1}=\lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{5}{x^2}}{5-\frac{4}{x}+\frac{1}{x^2}}=\frac{2+0+0}{5-0+0}=\frac{2}{5}.$$

(3) $$\lim_{x\to\infty}\frac{2x^2+1}{3x^3-2x^2+5x-2}=\lim_{x\to\infty}\frac{\frac{2}{x}+\frac{1}{x^3}}{3-\frac{2}{x}+\frac{5}{x^2}-\frac{2}{x^3}}=\frac{0+0}{3-0+0-0}=$$

(4) $\lim\limits_{x\to\infty} \frac{2x^3+2x-1}{x^2-5x+2} = \lim\limits_{x\to\infty} \frac{2+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}+\frac{2}{x^3}} = \frac{2+0-0}{0-0+0} = \frac{2}{0} = \infty$

That is the limit does not exist.

(5) $\lim\limits_{x\to\infty} \sqrt{x} = \lim\limits_{x\to+\infty} \sqrt{x} = +\infty$ or ∞.

(6) $\lim\limits_{x\to\infty} \left(2+\frac{\sin x}{x}\right) = \lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{\sin x}{x}$, but $\lim\limits_{x\to\infty} 2 = 2$

and $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$

because $-1 \leq \sin x \leq 1$ $\therefore \lim\limits_{x\to\infty} \left(2+\frac{\sin x}{x}\right) = 2+0 = 2$.

(7) $\lim\limits_{x\to-\infty} \left(2x+\frac{3}{x}\right) = -\infty + 0 = -\infty$.

(8) $\lim\limits_{x\to 2^-} \frac{1}{x^2-4} = \frac{1}{0} = -\infty$ and $\lim\limits_{x\to 2^+} \frac{1}{x^2-4} = \frac{1}{0} = +\infty$.

(9) $\lim\limits_{x\to\infty} (\sqrt{x^2+1} - x = \infty - \infty$ (meaning less).

$\lim\limits_{x\to\infty} (\sqrt{x^2+1} - x) \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim\limits_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$

$= \lim\limits_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$

$= \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{0}{\sqrt{1+0}+1} = \frac{0}{2} = 0$.

(10) $\lim\limits_{x\to\infty}(\sqrt{x^2+2x}-x)=\lim\limits_{x\to\infty}(\sqrt{x^2+2x}-x)\cdot\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}$

$=\lim\limits_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}=\lim\limits_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\lim\limits_{x\to\infty}\frac{2}{\sqrt{1+0}+1}=\frac{2}{2}=1$

More About An Asymptotes

Given $y=f(x)$. A line $y=mx+b$ is an asymptote for $f(x)$

(1) $m=\lim\limits_{x\to\infty}\frac{f(x)}{x}$ (2) $b=\lim\limits_{x\to\infty}(f(x)-mx)$

EX.(12) Find the asymptotes of the following functions:

(1) $y=f(x)=x+\frac{1}{x}=\frac{x^2+1}{x}$

$x=0$ is V. Asy.

$x^2+1=yx \Rightarrow x^2-yx+1=0 \Rightarrow x=\frac{y\pm\sqrt{y^2-4}}{2}$

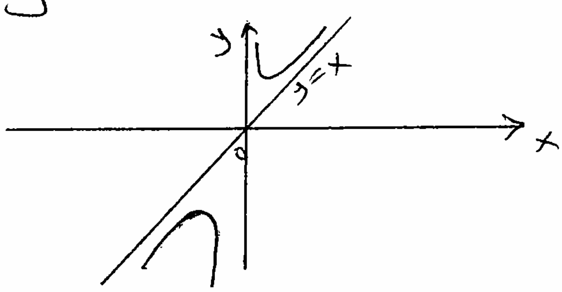
No H. Asy.

Let $y=mx+b$ be an asy.

$m=\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}\frac{\frac{x^2+1}{x}}{x}=\lim\limits_{x\to\infty}\frac{x^2+1}{x^2}=\lim\limits_{x\to\infty}1+\frac{1}{x^2}=1+0$
$=1$

$b=\lim\limits_{x\to\infty}(f(x)-mx)=\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x}-x\right)=\lim\limits_{x\to\infty}\frac{x^2+1-x^2}{x}$

$=\lim\limits_{x\to\infty}\frac{1}{x}=0$ $\therefore$ $y=x$ is an asy.



(2) $y = f(x) = \dfrac{x^2-3}{2x-4}$, $x = 2$ is V. Asy.

$x^2-3 = 2yx - 4y \Rightarrow x^2 - 2yx + 4y - 3 = 0$

$$x = \frac{4y \pm \sqrt{4y^2 - 4(4y-3)}}{2} = \frac{4y \pm \sqrt{4y^2-16y+12}}{2} = \frac{4y \pm 2\sqrt{y^2-4y+3}}{2}$$

$x = 2y \pm \sqrt{y^2-4y+3}$. No H. Asy.

Let $y = mx+b$ be an asymptote

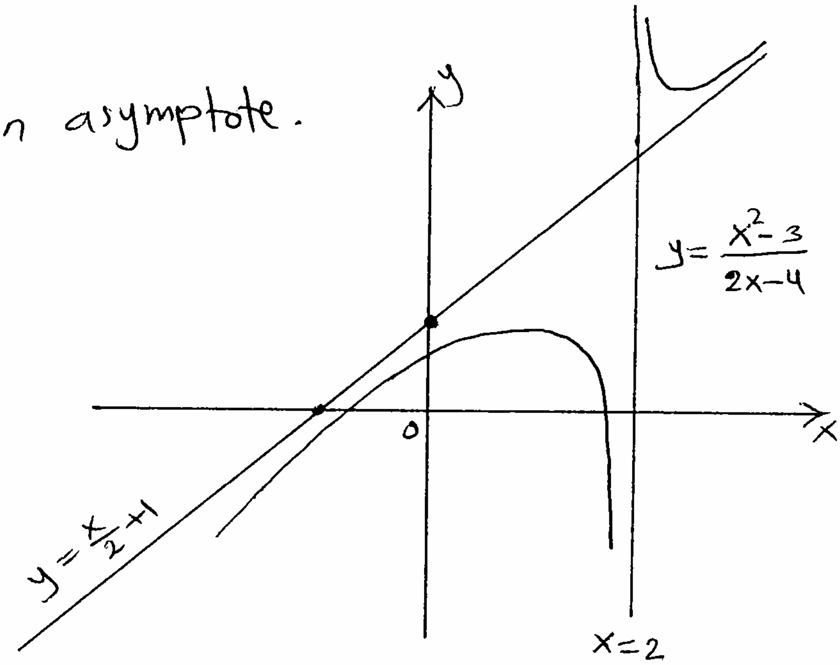
$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{x^2-3}{2x-4}}{x} = \lim_{x\to\infty} \frac{x^2-3}{2x^2-3x}$$

$$= \lim_{x\to\infty} \frac{1-\frac{3}{x^2}}{2-\frac{3}{x}} = \frac{1-0}{2-0} = \frac{1}{2}$$

$$b = \lim_{x\to\infty}(f(x)-mx) = \lim_{x\to\infty}\left(\frac{x^2-3}{2x-4} - \frac{1}{2}x\right) = \lim_{x\to\infty} \frac{x^2-3-x^2+2x}{2(x-2)}$$

$$= \lim_{x\to\infty} \frac{2x-3}{2x-4} = \lim_{x\to\infty} \frac{2-\frac{3}{x}}{2-\frac{4}{x}} = \frac{2-0}{2-0} = \frac{2}{2} = 1.$$

$\therefore$ $y = \frac{x}{2}+1$ is an asymptote.



<u>Sandwich Theorem</u> If $g(x) \le f(x) \le h(x)$ and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$.

Ex.(13) Find $\lim_{x \to \infty} f(x)$ if $\frac{2x+3}{x} \leq f(x) \leq \frac{2x^2+5x}{x^2}$

$$\lim_{x \to \infty} \frac{2x+3}{x} = \lim_{x \to \infty} \left(2 + \frac{3}{x}\right) = 2+0 = 2.$$

$$\lim_{x \to \infty} \frac{2x^2+5x}{x^2} = \lim_{x \to \infty} \left(2 + \frac{5}{x}\right) = 2+0 = 2.$$

$\therefore$ By Sandwich Theorem $\lim_{x \to \infty} f(x) = 2.$

Theorem(1) If θ is measured in radian, then

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Theorem(2) $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0.$

Proof: $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin\theta}{1+\cos\theta}$$

$$= (1) \cdot \frac{\sin(0)}{1+\cos(0)} = (1) \cdot \frac{0}{1+1} = \frac{0}{2} = 0.$$

Ex.(14) Find the following Limits :

(a) $\lim_{x \to 0} \frac{\sin 3x}{x} = 3 \lim_{3x \to 0} \frac{\sin 3x}{3x} = 3 \cdot (1) = 3$

as $x \to 0 \Rightarrow 3x \to 0.$

(b) $\lim_{x \to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x \to 0} \frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} = \frac{\lim_{x \to 0} \frac{\sin 5x}{x}}{\lim_{x \to 0} \frac{\sin 3x}{x}}$

as $x \to 0 \Rightarrow$ and $3x \to 0$, $5x \to 0$

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x} = \frac{5\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}}{3\lim\limits_{3x\to 0}\frac{\sin 3x}{3x}} = \frac{5(1)}{3(1)} = \frac{5}{3}.$$

(c) $$\lim_{x\to \pi/2}\frac{\cos x}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}}\frac{\sin(\frac{\pi}{2}-x)}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}}\frac{-\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}}$$

as $x\to\frac{\pi}{2} \Rightarrow x-\frac{\pi}{2}\to 0$

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}} = -\lim_{x-\frac{\pi}{2}\to 0}\frac{\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}} = -1.$$

(d) $$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}$$

$$= (1)\cdot\frac{1}{\cos(0)} = (1)\cdot\frac{1}{1} = 1.$$

(e) $$\lim_{x\to 0}\frac{\sin 2x}{2x^2+x} = \lim_{x\to 0}\frac{\sin 2x}{2x(x+\frac{1}{2})} = \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{x+\frac{1}{2}}$$

$$= \lim_{2x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{x+\frac{1}{2}} = (1)\cdot\frac{1}{0+\frac{1}{2}} = 2.$$

(f) $$\lim_{x\to\infty} x\sin\frac{1}{x}$$

Let $y=\frac{1}{x}$

as $x\to\infty \Rightarrow y=\frac{1}{x}\to 0$

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{y\to 0}\frac{\sin y}{y} = 1.$$

(g) $$\lim_{x\to 0}\frac{\sin x}{|x|}$$

$$\lim_{x\to 0^+}\frac{\sin x}{|x|} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1$$ and $$\lim_{x\to 0^-}\frac{\sin x}{|x|} = \lim_{x\to 0^-}\frac{\sin x}{-x} = -1$$

Since $\lim\limits_{x\to 0^+}\frac{\sin x}{|x|} \neq \lim\limits_{x\to 0^-}\frac{\sin x}{|x|}$, $\therefore \lim\limits_{x\to 0}\frac{\sin x}{|x|}$ does not exist

Defn. (Continuous function) الدالة المستمرة

A function $y=f(x)$ is said to be cont. at $x=a$ if

(1) $f(a)$ is define.

(2) $\lim\limits_{x \to a} f(x) = f(a)$.

Ex.(15)

(a) Every polynomial of the form

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is cont. ~~at x=a~~ for all x.

(b) $f(x) = \dfrac{1}{x}$

$f(x)$ is cont. for all x except at $x=0$, because $f(0)$ is not define.

(c) $f(x) = \dfrac{x+3}{(x-5)(x+2)}$, $f(x)$ is discont. at $x=5,-2$.

(d) $f(x) = \dfrac{\sin x}{x}$, $f(x)$ is discont. at $x=0$

(e) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$f(x)$ is cont. at $x=0$

(f) $f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$

$f(x)$ is cont. at $x=2$.

Derivatives الاشتقاقات

Defn. Let $y=f(x)$ be a function of x. If the limit

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ exist and define, we call it the derivative of f at x (or f is differentiable at x) and denoted by $\frac{dy}{dx}$, $\frac{df}{dx}$. That is

$$f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\ ,\quad \Delta x\neq 0$$

Rules of Derivatives : قواعد الاشتقاق

Rule(1) If $y=f(x)=C$, where c is constant, then

$$\frac{dy}{dx}=f'(x)=0 .$$

Rule(2) If is +ive integer and $y=f(x)=x^n$, then

$$f'(x)=\frac{dy}{dx}=n\,x^{n-1}$$

Rule(3) If $f(x)=c\,u(x)$, where c is constant and $u(x)$ is a differentiable of x then $f'(x)=c\,u'(x)$ or $\frac{dy}{dx}=c\frac{du}{dx}$.

Rule(4) If $u_i(x)$, $i=1,2,3,\dots,n$ are differentiable functions of x and $f(x)=u_1(x)\pm u_2(x)\pm\dots\pm u_n(x)$, then

$$f'(x)=u_1'(x)\pm u_2'(x)\pm\dots\pm u_n'(x) .$$

Rule(5) If $y=f(x)=u(x)\cdot v(x)$, where $u(x)$ and $v(x)$ are differentiable functions of x, then $f'(x)=u(x)\,v'(x)+v(x)\,u'(x)$.

or $$\frac{dy}{dx}=u(x)\cdot\frac{dv}{dx}+v(x)\cdot\frac{du}{dx} .$$

Rule (6) If $f(x) = \frac{u(x)}{v(x)}$, $v(x) \neq 0$ where $u(x)$ and $v(x)$ are differentiable functions of x then

$$f'(x) = \frac{v(x)\,u'(x) - u(x)\,v'(x)}{[v(x)]^2} \quad \text{or} \quad \frac{dy}{dx} = \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{[v(x)]^2}.$$

Rule (7) If $f(x) = [u(x)]^n$ where n is +ive integer and $u(x)$ is a differentiable function of x, then

$$f'(x) = n\,[u(x)]^{n-1}\,u'(x) \quad \text{or} \quad \frac{dy}{dx} = n\,[u(x)]^{n-1}\frac{du}{dx}.$$

Rule (8) If $f(x) = [u(x)]^n$ where n is −ive integer and $u(x)$ is a differentiable function of x then

$$f'(x) = n\,[u(x)]^{n-1}\,u'(x).$$

Rule (9) If $f(x) = [u(x)]^{\frac{p}{q}}$, where $u(x)$ is a differentiable function of x with p and q are integers $(q \neq 0)$, then

$$f'(x) = \frac{p}{q}[u(x)]^{\frac{p}{q}-1}\,u'(x).$$

Implicit Differentiations :

Consider the function defined by the eq. $f(x,y) = 0$ which may or may not be solved for y in terms of x.

for example $y - x^3 + 2x - 5 = 0$ Can be written as

$$y = x^3 - 2x + 5 \quad \text{and} \quad \frac{dy}{dx} = 3x^2 - 2.$$

While $y^5 + 4x^2y^2 + x^3 - 2 = 0$ can not be solved for y in terms of x.

Implicit differentiation enables us to find the derivative of such functions whenever they exist.

EX. Find $\frac{dy}{dx}$ if $y^3 - 3x^2y + x^3 = 5$

$$3y^2\frac{dy}{dx} - \left[3x^2\frac{dy}{dx} + y(6x)\right] + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} - 6xy + 3x^2 = 0$$

$$(3y^2 - 3x^2)\frac{dy}{dx} = 6xy - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3(2xy - x^2)}{3(y^2 - x^2} = \frac{2xy - x^2}{y^2 - x^2}$$

The second and Higher Derivatives :

Given the function $y = f(x)$. The derivative

$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}$ is the 1^{st} derivative of y w.r.to x

and

$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$ is Called the 2^{nd} derivative w.r.to x

Thus the second derivative is the derivative of the first derivative. That is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

In general, if $y = f(x)$ is a differentiable function of x, then the n^{th} derivative of y w.r.t. x is ~~defined~~ denoted by:

$$f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n}.$$

Ex. If $y=(3x^3+2x-1)^{1/2}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solu.

$$\frac{dy}{dx}=\frac{1}{2}(3x^3+2x-1)^{-\frac{1}{2}}(9x^2+2)$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[(3x^3+2x-1)^{-\frac{1}{2}}(18x)+(9x^2+2)\left[-\frac{1}{2}(3x^3+2x-1)^{-3/2}(9x^2+2)\right]\right]$$

Ex. If $y=3x^4-5x^3+6x-7$. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$

Solu.

$$\frac{dy}{dx}=12x^3-15x^2+6$$

$$\frac{d^2y}{dx^2}=36x^2-30x \Rightarrow \frac{d^3y}{dx^3}=72x-30 \Rightarrow \frac{d^4y}{dx^4}=72$$

$$\frac{d^5y}{dx^5}=\frac{d^6y}{dx^6}=\dots=\frac{d^ny}{dx^n}=0.$$

H.W

Ex. If $x^2-y^2=1$. Show that $\frac{dy}{dx}=\frac{x}{y}$ and $\frac{d^2y}{dx^2}=-\frac{1}{y^3}$.

Ex. If $y=(2x^2-5x^{-2})^{-5}$. Find $\frac{dy}{dx}$

$$\frac{dy}{dx}=-5(2x^2-5x^{-2})^{-6}(4x+10x^{-3}).$$

H.W

Ex. Find $\frac{d}{dx}\left[\frac{(x^2+x+1)^3}{(x^3+1)^4}\right]$

H.W

Ex. Find $\frac{dy}{dx}$, if $y=\frac{2x^3+3x-1}{x^2+1}$.

H.W

Ex. Find $\frac{dy}{dx}$, if $y=(x^2+2)(x^3+3x+1)$.

Chain Rule And Parametric Equations

Chain Rule If y is a function of x, say $y=f(x)$, and x is a function of t, say $x=g(t)$, then y is a function of t and

$$\boxed{\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}} \quad \cdots (1)$$

How we obtain eq.(1)

Since $y=f(x)$ and $x=g(t)$, then

$$y=f[g(t)] \text{ and } \frac{dy}{dt}=f'[g(t)]\cdot g'(t),$$

but $x=g(t)$ and $\frac{dx}{dt}=g'(t)$

$$\therefore \frac{dy}{dt}=f'(x)\cdot\frac{dx}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}.$$

Ex. If $y=x^3-2x^2+3$ and $x=t^2+2$. Find $\frac{dy}{dt}$ at $t=2$.

Solu.

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=(3x^2-4x)(2t)$$

when $t=2 \Rightarrow x=(2)^2+2=6$

$$\therefore \left.\frac{dy}{dt}\right|_{t=2}=[3(36)-4(6))=(108-24)(4)=336.$$

or $y=(t^2+2)^3-2(t^2+2)^2+3 \Rightarrow \frac{dy}{dt}=3(t^2+2)(2t)-4(t^2+2)(2t)$

$$\therefore \left.\frac{dy}{dt}\right|_{t=2}=3(36)(4)-4(6)(4)=336.$$

Parametric Equations

Sometimes, we may describe the curve by expressing both coordinates as functions of a third variable, say $x=g(t)$ and $y=f(t)$.

These two eq.'s are called the parametric eq.'s for x and y and the variable t is called a parameter.

Ex. Determine an equation in x and y of the following parametric eq.'s and then find $\frac{dy}{dx}$.

(a) $y=\frac{1}{t}$, $x=t \Rightarrow y=\frac{1}{x} \Rightarrow \frac{dy}{dx}=-\frac{1}{x^2}$.

(b) $y=t^2$, $x=\frac{t}{1-t}$

$x=\frac{t}{1-t} \Rightarrow x-xt=t \Rightarrow x=t+xt \Rightarrow x=t(1+x)$

$\Rightarrow t=\frac{x}{1+x} \quad \therefore y=\left(\frac{x}{x+1}\right)^2 \Rightarrow \frac{dy}{dx}=2\left(\frac{x}{x+1}\right)\left(\frac{x+1-x}{(x+1)^2}\right)$

$\therefore \frac{dy}{dx}=\frac{2x}{(x+1)^3}$.

(c) $y=2\sin t$, $x=3\cos t$

$\sin t=\frac{y}{2}$, $\cos t=\frac{x}{3} \Rightarrow \sin^2 t+\cos^2 t=\frac{y^2}{4}+\frac{x^2}{9}=1$

$\Rightarrow \frac{2y}{4}\frac{dy}{dx}+\frac{2x}{9}=0 \Rightarrow \frac{y}{2}\frac{dy}{dx}=-\frac{2}{9}x \Rightarrow \frac{dy}{dx}=-\frac{4}{9}\frac{x}{y}$.

(d) $y=\frac{5(4-t^2)}{4+t^2}$, $x=\frac{20t}{4+t^2}$

$y^2+x^2=\frac{25(4-t^2)^2}{(4+t^2)^2}+\frac{400t^2}{(4+t^2)^2}=\frac{25(16-8t^2+t^4)+400t^2}{(4+t^2)^2}$

$$= \frac{25(16-8t^2+t^4+16t^2)}{(4+t^2)^2} = \frac{25(16+8t^2+t^4)}{(4+t^2)^2}$$

$$= \frac{25(4+t^2)}{(4+t^2)^2}$$

$$\therefore\ y^2+x^2=25 \Rightarrow 2yy'+2x=0 \Rightarrow y'=\frac{-x}{y}.$$

H.W (e) $y=t^2$, $x=t-1$, H.W (f) $y=2+2\sin t$, $x=-1+2\cos t$.

H.W (g) $y=3\tan t$, $x=4\sec t$ H.W (h) $y=\sin^3 t$, $x=\cos^3 t$

H.W (i) $y=t^2+t-1$, $x=t^2+2t+3$ H.W (j) $y=\frac{3(2-t)(2+t)^2}{16t^2+8}$, $x=\frac{3(2-t)^2(2[illegible]}{6t^2+8}$

Derivatives of the Parametric Eq.'s

The 1st derivative

If $y=f(t)$ and $x=g(t)$, then

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dt}\cdot\frac{dt}{dx} \Rightarrow \boxed{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}}$$

Ex. If $y=t^2-1$ and $x=2t+3$. Find $\frac{dy}{dt}$, $\frac{dx}{dt}$ and $\frac{dy}{dx}$

Solu. $\frac{dy}{dt}=2t$, $\frac{dx}{dt}=2$, $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=(2t)\cdot\frac{1}{2}=t$

But $x=2t+3 \Rightarrow t=\frac{x-3}{2}$, $\therefore \frac{dy}{dx}=\frac{x-3}{2}$.

Ex. $y = t - \frac{1}{t}$, $x = t + \frac{1}{t}$

$$\frac{dy}{dt} = 1 + \frac{1}{t^2} , \quad \frac{dx}{dt} = 1 - \frac{1}{t^2} , \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{t^2+1}{t^2-1} \Rightarrow \frac{dy}{dx} = \frac{t + \frac{1}{t}}{t - \frac{1}{t}} = \frac{x}{y} .$$

or $y^2 - x^2 = (t - \frac{1}{t})^2 - (t + \frac{1}{t})^2 = \cancel{t^2} - 2 + \cancel{\frac{1}{t^2}} - \cancel{t^2} - 2 - \cancel{\frac{1}{t^2}}$

$$= -4$$

$$\therefore y^2 - x^2 = -4 \Rightarrow 2y \frac{dy}{dx} - 2x = 0 \Rightarrow 2y \frac{dy}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{x}{y} .$$

The 2nd derivative :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\frac{dx}{dt}} \cdot \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{1}{\frac{dx}{dt}} \cdot \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}}$$

Ex. If $x = t - t^2$, $y = t - t^3$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t=2$

Solu.

$$x = t - t^2 \Rightarrow \frac{dx}{dt} = 1 - 2t \Rightarrow \frac{d^2x}{dt^2} = -2 .$$

$$y = t - t^3 \Rightarrow \frac{dy}{dt} = 1 - 3t^2 \Rightarrow \frac{d^2y}{dt^2} = -6t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{1-12}{1-4} = \frac{11}{3} .$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^3} = \frac{-6t+12t^2+2-6t^2}{(1-2t)^3}$$

$$= \frac{6t^2-6t+2}{(1-2t)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{6(4)-6(2)+2}{(1-4)^3} = \frac{24-12+2}{(-3)^3} = \frac{14}{-27}$$

or $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-3t^2}{1-2t}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{1-3t^2}{1-2t}\right)$$

$$= \frac{\frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}}{(1-2t)}$$

$$= \frac{6t^2-6t+2}{(1-2t)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-2} = \frac{14}{-27}.$$

<u>Note</u> The chain Rule can be extended to many variables

In general: If $y=f_1(t_1)$, $t_1=f_2(t_2)$, $t_2=f_3(t_3)$, ...,

$t_{n-1}=f_n(t_n)$, $t_n=f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt_1}\cdot\frac{dt_1}{dt_2}\cdot\frac{dt_2}{dt_3}\cdot \cdots \cdot\frac{dt_{n-1}}{dt_n}\cdot\frac{dt_n}{dx}.$$

<u>EX.</u> If $y=x^3+2x^2+3x-4$. Find $\frac{dy^2}{dx^2}$.

<u>Solu.</u> Let $u=y^2$ and $v=x^2$

$$\frac{dy^2}{dx^2} = \frac{du}{dv} = \frac{du}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dv} = (2y)(3x^2+4x+3)\cdot\frac{1}{2x}$$

Ex. If $y=\frac{x}{x^2+1}$. Find $\frac{d\sqrt{y}}{d\sqrt{x}}$

Solu.

Let $u=\sqrt{y}$ and $v=\sqrt{x}$

$$\frac{d\sqrt{y}}{d\sqrt{x}}=\frac{du}{dv}=\frac{du}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dv}=\frac{1}{2\sqrt{y}}\cdot\left(\frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2}\right)\cdot 2\sqrt{x}$$

Ex. Find $\frac{d\sqrt{x^2+1}}{d\,x^3}$.

Solu.

Let $u=\sqrt{x^2+1}$ and $v=x^3$

$$\frac{du}{dv}=\frac{du}{dx}\cdot\frac{dx}{dv}=\frac{2x}{2\sqrt{x^2+1}}\cdot\frac{1}{3x^2}$$

Indeterminate forms :

The meaningless $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞, ∞^0, $0\cdot\infty$, $\infty-\infty$ are known as indeterminate forms.

Sometimes the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ produce $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when substituting $x=a$.

For example: $\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{4-4}{2-2}=\frac{0}{0}$ meaningless.

So the solution is

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=2+2=4.$$

L' Hopital's Rule (First form) قاعدة لوبيتال

Suppose $f(a)=g(a)=0$ or ∞ and $f'(a)$ and $g'(a)$ exist with $g'(a)\neq 0$. Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}$.

Ex. (a) $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2}\frac{2x}{1}=2(2)=4.$

(b) $\lim\limits_{x\to 0}\frac{x-2x^2}{3x^2+5x}=\lim\limits_{x\to 0}\frac{1-4x}{6x+5}=\frac{1-0}{0+5}=\frac{1}{5}$

(c) $\lim\limits_{x\to\infty}\frac{6x+5}{3x-8}=\lim\limits_{x\to\infty}\frac{6}{3}=\frac{6}{3}=2.$

(d) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim\limits_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-0}{1}=\lim\limits_{x\to 0}\frac{1}{2\sqrt{1+x}}=\frac{1}{2}$

L' Hopital's Rule (Stronger form):

Suppose $f(a)=g(a)=0$ or ∞ and the functions f and g with their derivatives are continuous in some interval I.

To find $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$, we proceed to differentiate f and g as long as we still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. But we stop the differentation as soon as one or the other derivatives is different from 0 or ∞ at $x=a$.

Note

L' Hopital's Rule does not apply when either numerator or denominator has a finite non-zero limit.

Ex: (a) $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{x^3-x^2-x+1} = \lim\limits_{x\to 1} \dfrac{3x^2-3}{3x^2-2x-1}$

$= \lim\limits_{x\to 1} \dfrac{6x}{6x-2} = \dfrac{6(1)}{6(1)-2} = \dfrac{6}{4} = \dfrac{3}{2}.$

(b) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2} = \lim\limits_{x\to 0} \dfrac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x}$

$= \lim\limits_{x\to 0} \dfrac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\dfrac{1}{8}.$

(c) $\lim\limits_{x\to \infty} \dfrac{2x^3-x^2+3x+1}{x^3+2x^2-x-1} = \lim\limits_{x\to \infty} \dfrac{6x^2-2x+3}{3x^2+4x-1}$

$= \lim\limits_{x\to \infty} \dfrac{12x-2}{6x+4} = \lim\limits_{x\to \infty} \dfrac{12}{6} = \dfrac{12}{6} = 2.$

"Transcendental Functions"

Trigonometric Functions الدوال المثلثية

مبرهنة

Theorem (1) If $y = f(x) = \sin x$ then $\frac{dy}{dx} = f'(x) = \cos x$.

برهان.

proof

Let $h = \Delta x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{-\sin x\,(1 - \cos h) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left[\frac{-\sin x (1 - \cos h)}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= -\sin x \lim_{h \to 0} \frac{1 - \cos h}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= -\sin x \cdot (0) + \cos x \cdot (1) = \cos x.$$

Theorem (2) If $y = f(x) = \cos x$, then $\frac{dy}{dx} = f'(x) = -\sin x$.

proof Let $h = \Delta x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \frac{-\cos x\,(1-\cos h)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= -\cos x \lim_{h\to 0} \frac{1-\cos h}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= -\cos x\,(0) - \sin(x)\,(1) = -\sin x .$$

Theorem (3)

(1) If $y=\tan x$ then $\frac{dy}{dx} = \sec^2 x$.

H.W
(2) If $y=\cot x$ then $\frac{dy}{dx} = -\csc^2 x$.

H.W
(3) If $y=\sec x$ then $\frac{dy}{dx} = \sec x \tan x$.

H.W
(4) If $y=\csc x$ then $\frac{dy}{dx} = -\csc x \cot x$.

proof (1)

$$y=\tan x = \frac{\sin x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$\therefore \frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x .$$

دالة قابلة للاشتقاق

Now, if $u=u(x)$ is a differentiable function of x and

1. $y=\sin u$ then $\frac{dy}{dx} = \cos u \cdot \frac{du}{dx}$.

2. $y=\cos u$ then $\frac{dy}{dx} = -\sin u \cdot \frac{du}{dx}$.

3. $y=\tan u$ then $\frac{dy}{dx} = \sec^2 u \cdot \frac{du}{dx}$.

4. $y=\cot u$ then $\frac{dy}{dx} = -\csc^2 u \cdot \frac{du}{dx}$.

5. $y=\sec u$ then $\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}$.

6. $y=\csc u$ then $\frac{dy}{dx} = -\csc u \cot u \frac{du}{dx}$.

EX(1) Find $\frac{dy}{dx}$ of the following:

(1) $y=\sin(x^2+2x-5) \Rightarrow \frac{dy}{dx}=\cos(x^2+2x-5)\cdot(2x+2)$

$=2(x+1)\cos(x^2+2x-5)$.

(2) $y=\sin^2(x^2+\frac{1}{x^2}) \Rightarrow \frac{dy}{dx}=2\sin(x^2+\frac{1}{x^2})\cdot\cos(x^2+\frac{1}{x^2})(2x-\frac{2}{x^3}$

(3) $y=\tan(2x)\cdot\cos(x^2+1) \Rightarrow$

$\frac{dy}{dx}=-\tan(2x)\sin(x^2+1)(2x)+\cos(x^2+1)\sec^2(2x)\cdot 2$.

(4) $y=\tan^{-3}(3x^2+\sec^2 2x)$

$\frac{dy}{dx}=-3\tan^{-4}(3x^2+\sec^2 2x)\cdot\sec^2(3x^2+\sec^2 2x).(6x+2\sec 2x$

$\sec 2x\tan 2x\cdot 2)$.

(5) $y=\frac{\sec[\sin(2x+1)]}{\tan(x^3+1)}$

$\frac{dy}{dx}=\frac{\tan(x^3+1)\sec[\sin(2x+1)].\tan[\sin(2x+1)].\cos(2x+1)\,2-\sec[\sin(2x+1)\cdot\sec^2(x^3+1).3x^2}{\tan^2(x^3+1)}$.

Ex.(2) Find $\frac{d}{dx}\left[\sec^{-2}(x^2+2x)-\tan^2(\sin 3x)\right]$

$=-2\sec^{-3}(x^2+2x)\sec(x^2+2x)\tan(x^2+2x)(2x+2)-2\tan(\sin 3x)$

$\sec^2(\sin 3x)\cdot\cos(3x)\cdot 3$.

EX.(3) Find $\frac{dy}{dx}$ if $x^2+5x-\tan^2(xy)=10$

Solu.

$$2x+5-2\tan(xy)\sec^2(xy)\cdot\left(x\frac{dy}{dx}+y\cdot(1)\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{2x+5-2y\tan(xy)\sec^2(xy)}{2x\tan(xy)\sec^2(xy)}$$

EX.(4) Find the eq. of tangent to the Curve $x\sin 2y=y\cos 2x$ at point $(\frac{\pi}{4},\frac{\pi}{2})$

Solu.

$$2x\cos 2y\cdot\frac{dy}{dx}+\sin 2y=-2y\sin 2x+\cos 2x\cdot\frac{dy}{dx}$$

$$2x\cos 2y\frac{dy}{dx}-\cos 2x\frac{dy}{dx}=-2y\sin 2x-\sin 2y$$

$$(2x\cos 2y-\cos 2x)\frac{dy}{dx}=-2y\sin 2x-\sin 2y$$

$$\therefore \frac{dy}{dx}=\frac{2y\sin 2x+\sin 2y}{\cos 2x-2x\cos 2y}$$ slope of tangent at any $P(x,y)$

$m=\frac{dy}{dx}$ at $x=\frac{\pi}{4}$, $y=\frac{\pi}{2}$

$$\text{is } m=\frac{2(\frac{\pi}{2})\sin\frac{\pi}{2}+\sin\pi}{\cos\frac{\pi}{2}-2(\frac{\pi}{4})\cos\pi}=\frac{\pi(1)+0}{0-\frac{\pi}{2}(-1)}$$

$$\therefore m=2 \Rightarrow m=\frac{y-y_1}{x-x_1} \Rightarrow 2=\frac{y-\frac{\pi}{2}}{x-\frac{\pi}{4}}$$ is the required eq.

EX.(5) If $f'(x)=\sin x^2$ and $y=f(\frac{2x+1}{x+1})$. Find $\frac{dy}{dx}$

Solu.

$$y=f\left(\frac{2x+1}{x+1}\right) \Rightarrow \frac{dy}{dx}=f'\left(\frac{2x+1}{x+1}\right)\cdot\frac{(x+1)(2)-(2x+1)(1)}{(x+1)^2}$$

$$\frac{dy}{dx}=\frac{1}{(x+1)^2}f'\left(\frac{2x+1}{x+1}\right)=\frac{1}{(x+1)^2}\sin\left(\frac{2x+1}{x+1}\right)^2.$$

EX.(6) If $y=\tan^{-3}(\sin 2x)$. Find $\frac{dy^2}{dx^2}$

Solu. Let $u=y^2$ and $v=x^2$

$$\frac{dy^2}{dx^2}=\frac{du}{dv}=\frac{du}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dv}$$

$$=2y\cdot\left[-3\tan^{-4}(\sin 2x)\sec^2(\sin 2x)\cos(2x).2\right]\cdot\frac{1}{2x}$$

EX.(7) If $y=\sec 2t$ and $x=\csc 2t$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{6}$

Solu.

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2\sec 2t\tan 2t}{-2\csc 2t\cot 2t}=-\tan^3 2t$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}}=-\tan^3\left(\frac{\pi}{3}\right)=-(\sqrt{3})^3=-3\sqrt{3}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}}{\frac{dx}{dt}}(-\tan^3 2t)=\frac{-3\tan^2 2t.\sec^2 2t.2}{-2\csc 2t\cot 2t}$$

$$=-3\tan^4 2t\sec 2t$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{6}}=-3\tan^4\left(\frac{\pi}{3}\right)\sec\left(\frac{\pi}{3}\right)=-3(\sqrt{3})^4.2=-54.$$

EX.(8) Evaluate the following Limits:

(1) $$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\frac{3\cos 3x}{5\cos 5x}=\frac{3\cos 0}{5\cos 0}=\frac{3\times 1}{5\times 1}=\frac{3}{5}.$$

(2) $$\lim_{x\to 0}\frac{\sin 2x}{2x^2+x}=\lim_{x\to 0}\frac{2\cos 2x}{4x+1}=\frac{2\cos 0}{0+1}=\frac{2\times 1}{1}=2.$$

③ $$\lim_{x\to 0}\frac{\tan 3x}{\sin x}=\lim_{x\to 0}\frac{3\sec^2 3x}{\cos x}=\lim_{x\to 0}\frac{3\sec^2 0}{\cos 0}$$
$$=\frac{3\times(1)^2}{1}=3.$$

④ $$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}$$

قاعدة اوبتال ٣ ثلاث مرات

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{\cos(0)}{6}=\frac{1}{6}$$

⑤ $$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty \quad \text{meaningless.}$$

قاعدة اوبتال مرتين

$$=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x\,(1)}$$

$$=\frac{1-\cos(0)}{0+\sin(0)}=\frac{1-1}{0+0}=\frac{0}{0}\quad\text{meaningless}$$

$$=\lim_{x\to 0}\frac{\sin x}{x(-\sin x)+\cos x\,(1)+\cos x}=\frac{\sin(0)}{0+\cos(0)+\cos(0)}=\frac{0}{0+1+1}$$
$$=\frac{0}{2}=0.$$

⑥ $$\lim_{x\to 0}\frac{\sin x^2}{x\sin x}=\frac{0}{0}\quad\text{meaningless}$$

$$=\lim_{x\to 0}\frac{\cos x^2\,(2x)}{x\cos x+\sin x\cdot(1)}=\frac{0}{0}\quad\text{meaningless}$$

$$=\lim_{x\to 0}\frac{\cos x^2\,(2)+2x\,(-\sin x^2\,(2x))}{-x\sin x+\cos x+\cos x}=\frac{2+0}{0+1+1}=\frac{2}{2}=1.$$

سؤال امتحاني

⑦ $$\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)=\infty-\infty$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin x}=\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}=\frac{0}{1}=0.$$

The Inverse of Trigonometric Functions

معكوس الدوال المثلثية

Defn.

(1) For $-1 \le x \le 1$, we define the no. $y = f(x) = \sin^{-1} x$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $x = \sin y$.

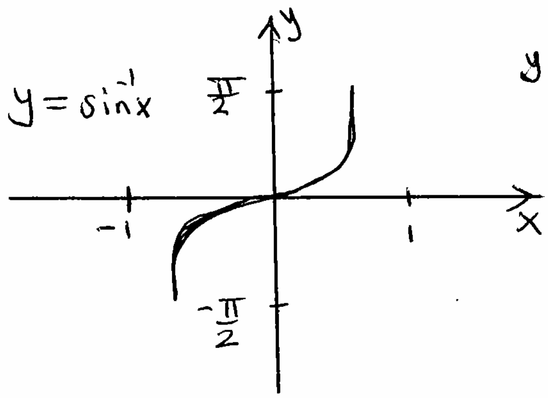
(2) For $-1 \le x \le 1$, we define the no. $y = f(x) = \cos^{-1} x$ for which $0 \le y \le \pi$ and $x = \cos y$.

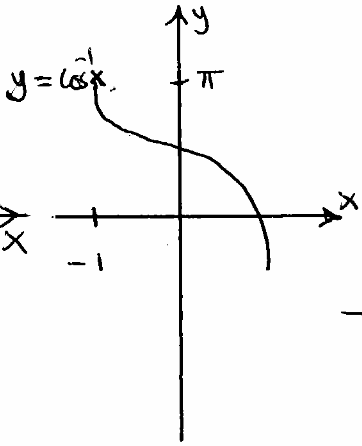
(3) For $-\infty < x < \infty$, we define the no. $y = f(x) = \tan^{-1} x$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $x = \tan y$.

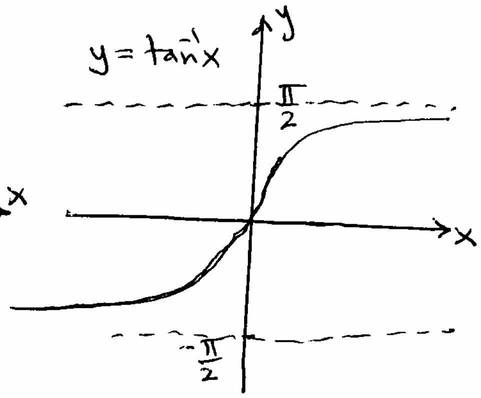
(4) For $-\infty < x < \infty$, we define the no. $y = f(x) = \cot^{-1} x$ for which $0 < y < \pi$ and $x = \cot y$.

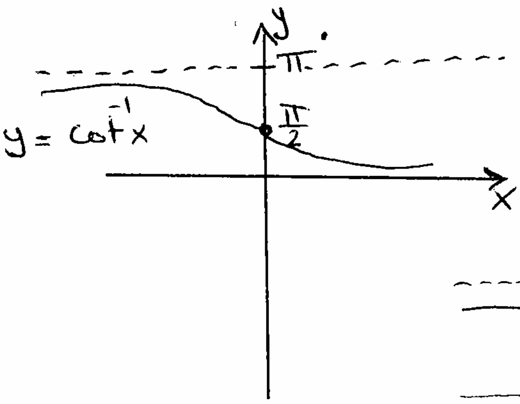
(5) For $x \le -1$ or $x \ge 1$, we define the no. $y = f(x) = \sec^{-1} x$ for which $-\pi \le y < -\frac{\pi}{2}$ or $0 \le y < \frac{\pi}{2}$ and $x = \sec y$.

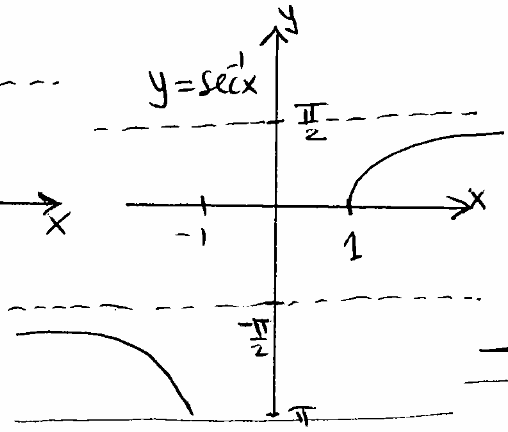
(6) For $x \le -1$ or $x \ge 1$, we define the no. $y = f(x) = \csc^{-1} x$ for which $-\pi < y \le -\frac{\pi}{2}$ or $0 < y \le \frac{\pi}{2}$ and $x = \csc y$.

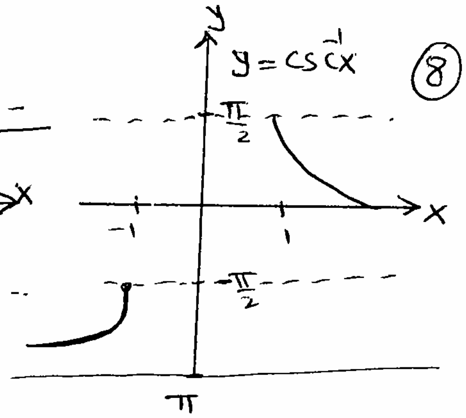












Note $\sin^{-1}x \neq \dfrac{1}{\sin x}$, $(\sin x)^{-1} = \dfrac{1}{\sin x} = \csc x$.

Some Important Properties of the Inverse of Trigonometric Functions

بعض الخواص المهمة لمعكوس الدوال المثلثية

(1) $\sin^{-1}(-x) = -\sin^{-1}x$.

(2) $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

(3) $\tan^{-1}(-x) = -\tan^{-1}x$.

(5) $\sec^{-1}(-x) = \pi - \sec^{-1}x$.

(4) $\cot^{-1}(-x) = -\cot^{-1}x$.

(6) $\csc^{-1}(-x) = -\csc^{-1}x$.

(7) $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$.

(8) $\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x$.

(9) $\sec^{-1}x = \frac{\pi}{2} - \csc^{-1}x$.

(10) $\sin^{-1}x = \csc^{-1}\frac{1}{x}$.

(11) $\cos^{-1}x = \sec^{-1}\frac{1}{x}$.

(12) $\tan^{-1}x = \cot^{-1}\frac{1}{x}$.

(13) $\sin^{-1}(\sin x) = x$

(14) $\sin(\sin^{-1}x) = x$.

proof (1)

Let $y = \sin^{-1}(-x) \Rightarrow -x = \sin y \Rightarrow x = -\sin y = \sin(-y)$

$\Rightarrow -y = \sin^{-1}x \Rightarrow y = -\sin^{-1}x \Rightarrow \sin^{-1}(-x) = -\sin^{-1}x$.

proof (2)

Let $y = \cos^{-1}(-x) \Rightarrow -x = \cos y \Rightarrow x = -\cos y \Rightarrow$

$x = \cos(\pi - y) \Rightarrow \pi - y = \cos^{-1}x \Rightarrow y = \pi - \cos^{-1}x$.

proof (3)

Let $y = \tan^{-1}(-x) \Rightarrow -x = \tan y \Rightarrow x = -\tan y$

$\Rightarrow x = \tan(-y) \Rightarrow -y = \tan^{-1}x \Rightarrow y = -\tan^{-1}x$.

proof (7)

Let $y = \sin^{-1}x \Rightarrow x = \sin y \Rightarrow x = \cos(\frac{\pi}{2} - y) \Rightarrow \frac{\pi}{2} - y = \cos^{-1}x$

$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}x$

proof (10)

Let $y = \sin^{-1}x \Rightarrow x = \sin y \Rightarrow x = \frac{1}{\csc y} \Rightarrow \frac{1}{x} = \csc y$

$\Rightarrow y = \csc^{-1}(\frac{1}{x})$.

proof (13)

Let $y = \sin^{-1}(\sin x) \Rightarrow \sin y = \sin x \Rightarrow x = y$.

EX.(1) If $\theta = \sin^{-1}\frac{\sqrt{3}}{2}$. find $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$.

solu. $\theta = \sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$

$\therefore \cos\frac{\pi}{3} = \cos\theta = \frac{1}{2}$ and so on ...

EX.(2) Evaluate

(1) $\sin(\cos^{-1}\frac{1}{\sqrt{2}})$ (2) $\sec(\cos^{-1}\frac{1}{2})$ (3) $\tan(\sin^{-1}(-\frac{1}{2}))$

(4) $\sin(\sin^{-1}\frac{\pi}{3})$.

solu.

1. Let $\theta = \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$

So $\sin(\cos^{-1}\frac{1}{\sqrt{2}}) = \sin\theta = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

2. Let $\theta = \cos^{-1}\frac{1}{2} \Rightarrow \frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$

So $\sec(\cos^{-1}\frac{1}{2}) = \sec(\theta) = \sec(\frac{\pi}{3}) = 2$.

3. Let $\theta = \sin^{-1}(-\frac{1}{2}) \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}$

So $\tan(\sin^{-1}(-\frac{1}{2})) = \tan(\theta) = \tan(-\frac{\pi}{6}) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$.

4. $\sin(\sin^{-1}\frac{\pi}{3}) = \frac{\pi}{3}$.

EX.(3) Solve for x if $\tan^{-1}x - \cot^{-1}x = \frac{\pi}{4}$

Solu.

$$\tan^{-1}x - (\frac{\pi}{2} - \tan^{-1}x) = \frac{\pi}{4} \Rightarrow \tan^{-1}x - \frac{\pi}{2} + \tan^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow 2\tan^{-1}x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

$$\Rightarrow \tan^{-1}x = \frac{3\pi}{8} \Rightarrow x = \tan\frac{3\pi}{8}$$

$$x = \tan\frac{3\pi}{8} = \tan\left(\frac{\frac{3\pi}{4}}{2}\right) = \frac{\sin\frac{3\pi}{4}}{1+\cos\frac{3\pi}{4}} = \frac{\sin(\frac{\pi}{2}+\frac{\pi}{4})}{1+\cos(\frac{\pi}{2}+\frac{\pi}{4})}$$

$$= \frac{\cos\frac{\pi}{4}}{1-\sin\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1}.$$

ملاحظة

$$\boxed{\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}}$$

Derivative of The Inverse of Trigonometric Functions

مشتقة معكوس الدوال المثلثية

Theorem:

(1) If $y = \sin^{-1}x$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

(2) If $y = \cos^{-1}x$ then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

(3) If $y = \tan^{-1}x$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$

(4) If $y = \cot^{-1}x$ then $\frac{dy}{dx} = \frac{-1}{1+x^2}$

(5) If $y = \sec^{-1}x$ then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$.

(6) If $y=\csc^{-1}x$ then $\frac{dy}{dx}=\frac{-1}{x\sqrt{x^2-1}}$

Proof (1)

$$y=\sin^{-1}x \Rightarrow x=\sin y \Rightarrow 1=\cos y\,\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}.$$

Proof (2)

$$y=\cos^{-1}x \Rightarrow x=\cos y \Rightarrow 1=-\sin y\,\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}=\frac{-1}{\sqrt{1-x^2}}.$$

Proof (3)

$$y=\tan^{-1}x \Rightarrow x=\tan y \Rightarrow 1=\sec^2 y\,\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

Now, if $u=u(x)$ is a differentiable function of x and

1. $y=\sin^{-1}u$ then $\frac{dy}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$.
2. $y=\cos^{-1}u$ then $\frac{dy}{dx}=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$.
3. $y=\tan^{-1}u$ then $\frac{dy}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$.
4. $y=\cot^{-1}u$ then $\frac{dy}{dx}=\frac{-1}{1+u^2}\frac{du}{dx}$.
5. $y=\sec^{-1}u$ then $\frac{dy}{dx}=\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$.
6. $y=\csc^{-1}u$ then $\frac{dy}{dx}=\frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$.

EX.(4) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = \sin^{-1}(x^2+3x-1) \Rightarrow \frac{dy}{dx} = \frac{(2x+3)}{\sqrt{1-(x^2+3x-1)^2}}$

(2) $y = x^2 \tan^{-1}\sqrt{x} \Rightarrow \frac{dy}{dx} = x^2 \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} + 2x \tan^{-1}\sqrt{x}$.

(3) $y = \cos^{-1}(x^2+\tan^{-1}3x) \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2+\tan^{-1}3x)}} \cdot (2x + \frac{3}{1+9x^2})$

(4) $y = \sin^2(\sec^{-1}2x) \cdot \cot^{-1}(\frac{1}{x})$

$\frac{dy}{dx} = \sin^2(\sec^{-1}2x) \cdot \frac{-1}{1+(\frac{1}{x})^2} \cdot (-\frac{1}{x^2}) + \cot^{-1}(\frac{1}{x}) \cdot 2\sin(\sec^{-1}2x)$

$\cos(\sec^{-1}2x) \cdot \frac{2}{2x\sqrt{4x^2-1}}$.

EX.(5) Find $\frac{d}{dx}\left[\frac{\tan^2(3x^2+1)}{\sin^{-1}(x^2-1)}\right]$

$$= \frac{\sin^{-1}(x^2-1) \cdot 2\tan(3x^2+1)\sec^2(3x^2+1)(6x) - \tan^2(3x^2+1)\frac{(2x)}{\sqrt{1-(x^2-1)^2}}}{[\sin^{-1}(x^2-1)]^2}$$

EX.(6) If $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$. Find $\frac{dy^3}{d\sec 2x}$

solu.

Let $u = y^3 \Rightarrow \frac{du}{dy} = 3y^2$

$v = \sec 2x \Rightarrow \frac{dv}{dx} = 2\sec 2x \tan 2x$

$$\frac{dy^3}{d\sec 2x} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv} = 3y^2 \cdot \frac{\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}}{\sqrt{1-(\frac{1-x}{1+x})^2}} \cdot \frac{1}{2\sec 2x \tan 2x}$$

EX.(7) If $y=\sin^{-1}t$ and $x=\cos^{-1}t$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t=\frac{1}{2}$

Solu.

$$y=\sin^{-1}t \Rightarrow \frac{dy}{dt}=\frac{1}{\sqrt{1-t^2}} \quad , \quad x=\cos^{-1}t \Rightarrow \frac{dx}{dt}=\frac{-1}{\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-1 \quad \text{and} \quad \frac{d^2y}{dx^2}=0.$$

H.W

EX.(8) Show that the functions

$$f(x)=\sin^{-1}\left(\frac{x-1}{x+1}\right) \text{ and } g(x)=2\tan^{-1}\sqrt{x}$$

have the same derivative.

EX.(9): Evaluate the following Limits:

(1) $$\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}=1.$$

(2) $$\lim_{x\to 0}\frac{2\tan^{-1}3x}{5x}=\lim_{x\to 0}\frac{2\,\frac{3}{1+(3x)^2}}{5}=\frac{6}{5}.$$

(3) $$\lim_{x\to 0}\frac{\tan^{-1}x-x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{3x^2}=\lim_{x\to 0}\frac{\frac{(1+x^2)(0)-1(2x)}{(1+x^2)^2}}{6x}$$

$$=\lim_{x\to 0}\frac{\frac{-2x}{(1+x^2)^2}}{6x}=\lim_{x\to 0}\frac{\frac{(1+x^2)^2(-2)-(-2x)\,2(1+x^2)(2x)}{(1+x^2)^4}}{6}$$

$$=\frac{\frac{-2-0}{1}}{6}=\frac{-2}{6}=-\frac{1}{3}.$$

H.W

(4) $$\lim_{x\to 0}\frac{\sin^{-1}x-x}{x^3}=\frac{1}{6}$$ استعمال طريقة لوبيتال ثلاث مرات .

The Logarithmic Function الدالة اللوغارتمية

The logarithm was discovered by a Scottish Nobleman John Napier (1550 – 1617)

$$y = f(x) = \log_b x \iff x = b^y$$

where y is the logarithm
x is the number
b is the base

If $b = 10$, we write $y = \log_{10} x$ or $y = \log x$

If $b = e = 2.7183$, we write $y = \log_e x$ or $y = \ln x$

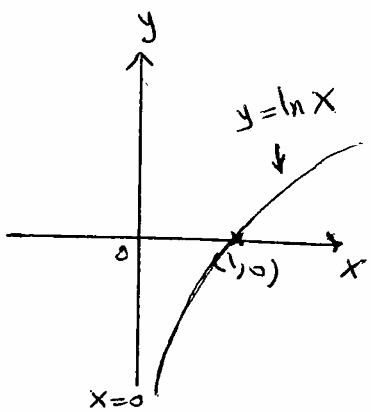
$\ln$ read natural Logarithm (اللوغاريتم الطبيعي)

Relation Between The Logarithm and The Natural Logarithm

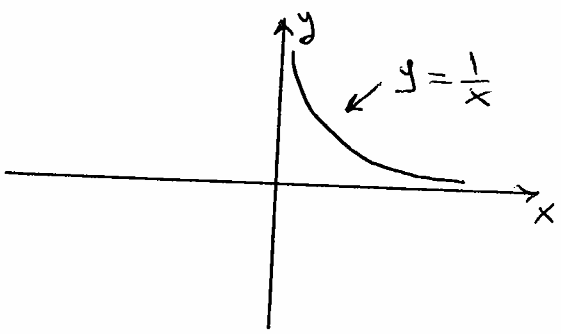
Let $y = \log_b x \iff x = b^y \Rightarrow \ln x = \ln b^y = y \ln b$

$$\Rightarrow y = \frac{\ln x}{\ln b}$$

So $\log_b x = \dfrac{\ln x}{\ln b}$



Defn.: For $x > 0$, we define $\ln x = \displaystyle\int_1^x \frac{dt}{t}$



Properties

1. $\ln(a \cdot b) = \ln a + \ln b$

2. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

3. $\ln 1 = 0$

4. $\ln a^r = r \ln a$ where $r = \frac{p}{q}$, p and $q \neq 0$ are integers.

proof (1)

$$\ln(a.b) = \int_1^{a.b} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{a.b} \frac{dt}{t} = \ln a + \int_a^{a.b} \frac{dt}{t}$$

Let $u = \frac{t}{a} \Rightarrow au = t \Rightarrow a\,du = dt$

$$\int_a^{a.b} \frac{dt}{t} = \int_1^{b} \frac{a\,du}{a t} = \int_1^{b} \frac{du}{u} = \ln(b)$$

$\therefore \quad \ln(a.b) = \ln a + \ln b$

proof (2)

$$a = \frac{a}{b} \cdot b \Rightarrow \ln(a) = \ln\left(\frac{a}{b} \cdot b\right) = \ln\left(\frac{a}{b}\right) + \ln(b)$$

$$\Rightarrow \quad \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b).$$

proof (3)

$$1 = \frac{a}{a} \Rightarrow \ln 1 = \ln\left(\frac{a}{a}\right) = \ln a - \ln a = 0$$

proof (4) Let $u = a^{\frac{1}{q}}$ then $a^r = a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = u^p$.

Also $a^r = u^p \Rightarrow \ln a^r = \ln u^p = p \ln u \quad \sim (1)$

$u = a^{\frac{1}{q}} \Rightarrow a = u^q \Rightarrow \ln a = q \ln u \Rightarrow \frac{1}{q} \ln a = \ln u \sim (2)$

From (1) and (2)

$$\ln a^r = p \cdot \frac{1}{q} \ln a = \frac{p}{q} \ln a = r \ln a.$$

Derivative of The Natural Logarithm

مشتقة دالة اللوغاريتم الطبيعي

Theorem

If $y = f(x) = \ln x$ then $\dfrac{dy}{dx} = f'(x) = \dfrac{1}{x}$

proof: $\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln x) = \dfrac{d}{dx}\displaystyle\int_1^x \dfrac{dt}{t} = \dfrac{1}{x}$ (by Fundamental Theorem of Calculus)

Now, If $u = u(x)$ is a differential function of x and

$y = \ln u$ then $\dfrac{dy}{dx} = \dfrac{1}{u} \cdot \dfrac{du}{dx}$

EX.(1) Find $\dfrac{dy}{dx}$ of the following Functions:

(1) $y = \ln(x^3 + 2x^2 - 3x + 5) \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 3}{(x^3 + 2x^2 - 3x + 5)}$

(2) $y = \ln(x^{-2} + \sin^2 3x) \Rightarrow \dfrac{dy}{dx} = \dfrac{(-2x^{-3} + 6\sin 3x \cos 3x)}{(x^{-2} + \sin^2 3x)}$

(3) $y = \sin^{-1}(\ln x) . \ln(\sin^{-1} 3x)$

$$\frac{dy}{dx} = \sin^{-1}(\ln x) . \frac{\frac{3}{\sqrt{1-(3x)^2}}}{\sin^{-1} 3x} + \ln(\sin^{-1} 3x) . \frac{\frac{1}{x}}{\sqrt{1-(\ln x)^2}}$$

(4) $y = \ln[\ln(\sec^2 2x + x\sin^{-1} x)]$

$$\frac{dy}{dx} = \frac{1}{\ln(\sec^2 2x + x\sin^{-1} x)} \cdot \frac{1}{\sec^2 2x + x\sin^{-1} x} \cdot$$

$$\left(2\sec 2x \cdot \sec 2x \tan 2x \cdot 2 + \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right).$$

H.W

(5) $y = \dfrac{x + \ln(\sec 3x)}{\ln(x + \frac{1}{x})}$.

EX.(2) If $y = \dfrac{x^{3/2}(3x+2)^{1/2}(x^2+3x-1)^{4/3}}{(x^3+2)^{5/2}}$. Find $\dfrac{dy}{dx}$

Solu.

$$\ln y = \frac{3}{2}\ln x + \frac{1}{2}\ln(3x+2) + \frac{4}{3}\ln(x^2+3x-1) - \frac{5}{2}\ln(x^3+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{3}{2}\,\frac{1}{x} + \frac{1}{2}\,\frac{3}{(3x+2)} + \frac{4}{3}\,\frac{(2x+3)}{(x^2+3x-1)} - \frac{5}{2}\,\frac{3x^2}{(x^3+2)}\right]$$

$$\therefore \frac{dy}{dx} = y\left[\quad ---- \quad\right]$$

EX.(3) If $y = \dfrac{(x^2+1)^{3/2}(x^2-1)^{-3/2}\tan^{-1}(\sin 2x)}{(x^2+4)^{2/3}\sin^{-3}2x}$. Find $\dfrac{dy}{dx}$

Solu.

$$\ln y = \frac{3}{2}\ln(x^2+1) - \frac{3}{2}\ln(x^2-1) + \ln[\tan^{-1}(\sin 2x)] - \frac{2}{3}\ln(x^2+4)$$
$$+3\ln(\sin 2x)$$

$$\frac{1}{y}y' = \left[\frac{3}{2}\,\frac{2x}{(x^2+1)} - \frac{3}{2}\,\frac{2x}{(x^2-1)} + \frac{\frac{2\cos 2x}{1+\sin^2 2x}}{\tan^{-1}(\sin 2x)} - \frac{2}{3}\,\frac{2x}{(x^2+4)}\right.$$
$$\left.+6\,\frac{\cos 2x}{\sin 2x}\right]$$

$$\therefore y' = y\left[\quad ----- \quad\right]$$

EX.(4) Find $\dfrac{dy}{dx}$ of the following functions :

① $y = x^{\sin x}$

solu. $\ln y = \sin x \ln x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{\sin x}{x} + \cos x \ln x$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right].$$

② $y = (\ln x)^x \Rightarrow \ln y = x \ln(\ln x)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \frac{1}{\ln x} \cdot \frac{1}{x} + \ln(\ln x) \cdot (1)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right] = (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right].$$

③ $y = (\tan^{-1}x)^{\frac{x\sin x}{x^2+1}} \Rightarrow \ln y = \frac{x \sin x}{x^2+1} \cdot \ln(\tan^{-1}x)$

$$\Rightarrow \ln(\ln y) = \ln x + \ln \sin x - \ln(x^2+1) + \ln[\ln(\tan^{-1}x)]$$

$$\frac{1}{\ln y} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{2x}{(x^2+1)} + \frac{1}{\ln(\tan^{-1}x)} \cdot \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = y \ln y \left[\frac{1}{x} + \cot x - \frac{2x}{(x^2+1)} + \frac{1}{(1+x^2)\ln(\tan^{-1}x)}\right]$$

EX.(5) Evaluate the following Limits:

(1) $\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0.$

(2) $\lim_{x \to 0} \frac{\ln(1+2x) - 2x}{x^2} = \lim_{x \to 0} \frac{\frac{2}{1+2x} - 2}{2x} = \lim_{x \to 0} \frac{-\frac{4}{(1+2x)^2}}{2} = -2$

(3) $\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{1}{\ln x} = 0.$

EX.(6) Solve for x if $3^x = 2^{x+1}$

Solu. Take Logarithm of both side

$$x \ln 3 = (x+1)\ln 2 \Rightarrow x \ln 3 = x \ln 2 + \ln 2$$

$$x \ln 3 - x \ln 2 = \ln 2 \Rightarrow x(\ln 3 - \ln 2) = \ln 2$$

$$\therefore x = \frac{\ln 2}{\ln 3 - \ln 2}.$$

The Exponential Function الدالة الأسية

<u>Defn.</u> The exponential function is defined as an inverse of the natural logarithm, and denoted by exp or e.

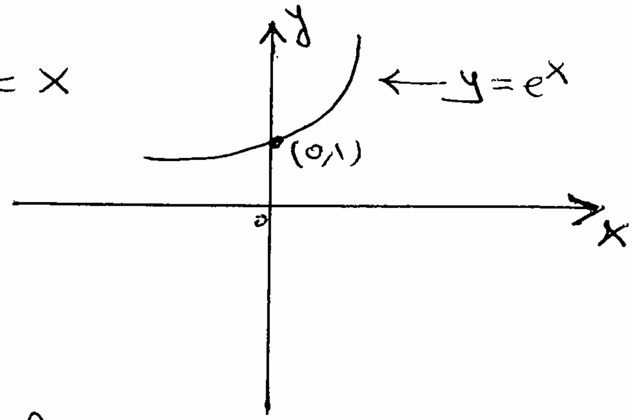
That is

For $-\infty < x < \infty$, we define $y = f(x) = e^x \iff x = \ln y$, $0 < y < \infty$.

<u>properties</u>

(1) $e = 2.7183$ (2) $e^{x+y} = e^x \cdot e^y$ (3) $e^{x-y} = \frac{e^x}{e^y}$

(4) $e^{\ln x} = x$ (5) $\ln e^x = x$



<u>proof (2)</u>

Let $u = e^x \Rightarrow x = \ln u$
Let $v = e^y \Rightarrow y = \ln v$
$x + y = \ln u + \ln v \Rightarrow x + y = \ln(u \cdot v)$
$\Rightarrow u \cdot v = e^{x+y} \Rightarrow e^x \cdot e^y = e^{x+y}$.

<u>proof (4)</u>

Let $y = e^{\ln x} \Rightarrow \ln x = \ln y \Rightarrow x = y = e^{\ln x}$.

<u>proof (5)</u>

Let $y = \ln e^x \Rightarrow e^y = e^x \Rightarrow x = y = \ln e^x$.

<u>EX.(1)</u> Simplify the following expressions:
بسط التعابير الآتية :

1. $e^{\ln 2} = 2$ 2. $e^{\ln(x^2+1)} = x^2 + 1$

3. $\ln e^{-1.3} = -1.3$ 4. $\ln e^{\sin x} = \sin x$

5. $\ln\left(\frac{e^{2x}}{5}\right) = \ln e^{2x} - \ln 5 = 2x - \ln 5$.

6. $e^{\ln 2 + 3\ln x} = e^{\ln 2} \cdot e^{3\ln x} = 2 \cdot e^{\ln x^3} = 2x^3$.

7. $e^{2x+\ln x} = e^{2x} \cdot e^{\ln x} = x e^{2x}$.

EX.(2) Solve for y if

(1) $\ln(y-1) - \ln y = 2x$

Solu. $\ln(y-1) - \ln y = 2x \Rightarrow \ln\left(\frac{y-1}{y}\right) = 2x$

$\Rightarrow \frac{y-1}{y} = e^{2x} \Rightarrow y - 1 = y e^{2x} \Rightarrow y(1 - e^{2x}) = 1$

$\therefore y = \frac{1}{1-e^{2x}}$

(2) $\ln(y-1) = x \ln x \Rightarrow y - 1 = e^{x+\ln x}$
$= e^x \cdot e^{\ln x} = x e^x$

$\Rightarrow y = x e^x + 1$.

Derivative of The Exponential Function مشتقة الدالة الأسية

Theorem

If $y = e^x$ then $\frac{dy}{dx} = e^x$

Proof

$y = e^x \Rightarrow x = \ln y \Rightarrow 1 = \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y = e^x$.

Now, if $u(x) = u$ is a differentiable function of x and $y = e^u$ then $\frac{dy}{dx} = e^u \cdot \frac{du}{dx}$

EX.(1) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = e^{x^2+\sin 2x} \Rightarrow \frac{dy}{dx} = e^{x^2+\sin 2x} \cdot (2x + 2\cos 2x)$.

(2) $y = e^{\tan^{-1}2x + \ln x} = x\, e^{\tan^{-1}2x} \Rightarrow \frac{dy}{dx} = x\, e^{\tan^{-1}2x} \cdot \frac{2}{1+(2x)^2} + e^{\tan^{-1}2x} \cdot (1)$

$\therefore \frac{dy}{dx} = \left(\frac{2x}{1+4x^2} + 1\right) e^{\tan^{-1}2x}$.

(3) $y = \tan^{-1}(e^{2x}) \Rightarrow \frac{dy}{dx} = \frac{2e^{2x}}{1+e^{4x}}$

(4) $y = e^{\sec x} \cdot \sec e^x \Rightarrow \frac{dy}{dx} = e^{\sec x} \cdot \sec(e^x) . \tan(e^x)\, e^x$

$+ \sec(e^x) . e^{\sec x} \sec x \tan x$

Ex(2) If $y = (\sin x)^{e^x}$, Find $\frac{d\, e^{\sin^{-1}y}}{d \ln x}$

Solu.

Let $u = e^{\sin^{-1}y}$ and $v = \ln x$

$\frac{d\, e^{\sin^{-1}y}}{d \ln x} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$

$u = e^{\sin^{-1}y} \Rightarrow \frac{du}{dy} = e^{\sin^{-1}y} \cdot \frac{1}{\sqrt{1-y^2}}$, $v = \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$

$y = (\sin x)^{e^x} \Rightarrow \ln y = e^x \ln(\sin x)$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \left[e^x \cdot \frac{\cos x}{\sin x} + \ln(\sin x)\, e^x\right]$

$\therefore \frac{dy}{dx} = y\left[e^x \cot x + e^x \ln(\sin x)\right]$

$\therefore \frac{dy}{dx} = (\sin x)^{e^x}\left[e^x \cot x + e^x \ln(\sin x)\right]$.

$\Rightarrow \frac{d\, e^{\sin^{-1}y}}{d \ln x} = \frac{e^{\sin^{-1}y}}{\sqrt{1-y^2}} \cdot (\sin x)^{e^x}\left[e^x \cot x + e^x \ln(\sin x)\right] . x$.

where $y = (\sin x)^{e^x}$.

EX.(3) Evaluate the following Limits :

(1) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^3} = \lim\limits_{x\to\infty} \dfrac{e^x}{3x^2} = \lim\limits_{x\to\infty} \dfrac{e^x}{6x} = \lim\limits_{x\to\infty} \dfrac{e^x}{6}$

$= \dfrac{e^\infty}{6} = \infty$.

(2) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^x} = \lim\limits_{x\to\infty} \dfrac{3x^2}{e^x} = \lim\limits_{x\to\infty} \dfrac{6x}{e^x} = \lim\limits_{x\to\infty} \dfrac{6}{e^x} = \dfrac{6}{e^\infty} = \dfrac{6}{\infty} = 0$

(3) $\lim\limits_{x\to 0} \dfrac{e^{2x}-2x-1}{1-\cos x} = \lim\limits_{x\to 0} \dfrac{2e^{2x}-2}{\sin x} = \lim\limits_{x\to 0} \dfrac{4e^{2x}}{\cos x} = \dfrac{4e^0}{\cos 0} = \dfrac{4}{1} = 4$

(4) $\lim\limits_{x\to 0^+} x^{\frac{1}{x}} = \lim\limits_{x\to 0^+} e^{\ln x^{\frac{1}{x}}} = \lim\limits_{x\to 0^+} e^{\frac{1}{x}\ln x} = e^{\lim\limits_{x\to 0^+} \frac{\ln x}{x}}$

$= e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{1}} = e^{\frac{\infty}{1}} = e^\infty = \infty$

(5) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = \lim\limits_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = e^{\lim\limits_{x\to 0} \frac{\ln(1+x)}{x}} = e^{\lim\limits_{x\to 0} \frac{1}{1+x}}$

$= e^{\frac{1}{1+0}} = e^1 = e$.

H.W

(6) $\lim\limits_{x\to\infty} \left(1+\dfrac{b}{x}\right)^{cx} = e^{bc}$.

H.W

EX.(4) Solve for x if $3y = \dfrac{e^{2x}-1}{e^{2x}+1}$

H.W

EX.(5) $\lim\limits_{x\to 4} \dfrac{e^{x-4}-x+4}{\cos^2(\pi x)}$.

The Function a^x

Defn. For $a>0$, we define $a^x = e^{x \ln a}$

Theorem If $y = a^x$ then $\frac{dy}{dx} = a^x . \ln a$

proof

$$y = a^x = e^{x \ln a} \Rightarrow \frac{dy}{dx} = e^{x \ln a} . \ln a$$
$$= a^x . \ln a .$$

Now, if $u = u(x)$ is a differentiable function of x and

$y = a^u$ then $\frac{dy}{dx} = a^u . \ln a . \frac{du}{dx}$

EX.(1) Find $\frac{dy}{dx}$ of the following functions:

① $y = 2^{\sin^2 2x} \Rightarrow \frac{dy}{dx} = 2^{\sin^2 2x} . \ln 2 . (2 \sin 2x \cos 2x . 2)$

$$= 2^{\sin^2 2x} . \ln 2 . (4 \sin 2x \cos 2x)$$

② $y = 3^{\tan^{-1} 2x} . \ln(\sec 2x)$.

$$y' = \frac{dy}{dx} = 3^{\tan^{-1} 2x} . \frac{2 \sec 2x . \tan 2x}{\sec 2x} + \ln(\sec 2x) . 3^{\tan^{-1} 2x} . \ln 3 \left(\frac{2}{1+4x^2} \right)$$

$$= 2 . 3^{\tan^{-1} 2x} . \tan(2x) + \ln 3 . \ln(\sec 2x) . 3^{\tan^{-1} 2x} . \frac{2}{1+4x^2} .$$

EX.(2) Find the following Limits:

(1) $\lim_{x \to \infty} 2^{-x} = 2^{-\infty} = 0$ (2) $\lim_{x \to -\infty} 3^x = 3^{-\infty} = 0$.

(3) $\lim_{x \to 0} \frac{3^{\sin x} - 1}{x} = \lim_{x \to 0} \frac{3^{\sin x} \ln 3 \cos x}{1} = 3^0 . \ln 3 . \cos 0 = (1) . \ln 3 . (1)$
$= \ln 3$.

(4) $$\lim_{x \to 0} \frac{3^x - 2^x}{x} = \lim_{x \to 0} \frac{3^x \ln 3 - 2^x \ln 2}{1}$$

$$= 3^0 \ln 3 - 2^0 \ln 2 = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right).$$

EX.(3) Solve for x if $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$

Solu.

Since $$3^{\log_3 7} = e^{\log_3 7 \cdot \ln 3} = e^{\frac{\ln 7}{\ln 3} \cdot \ln 3} = e^{\ln 7} = 7.$$

And $$2^{\log_2 5} = e^{\log_2 5 \cdot \ln 2} = e^{\frac{\ln 5}{\ln 2} \cdot \ln 2} = e^{\ln 5} = 5$$

So $$7 + 5 = x \Rightarrow x = 12.$$

The Hyperbolic Functions الدوال الزائدية

تراكيب خاصة

The hyperbolic functions are a special combinations of the functions e^x and e^{-x}.

الدوال الزائدية تراكيب خاصة من الدوال e^x, e^{-x}.

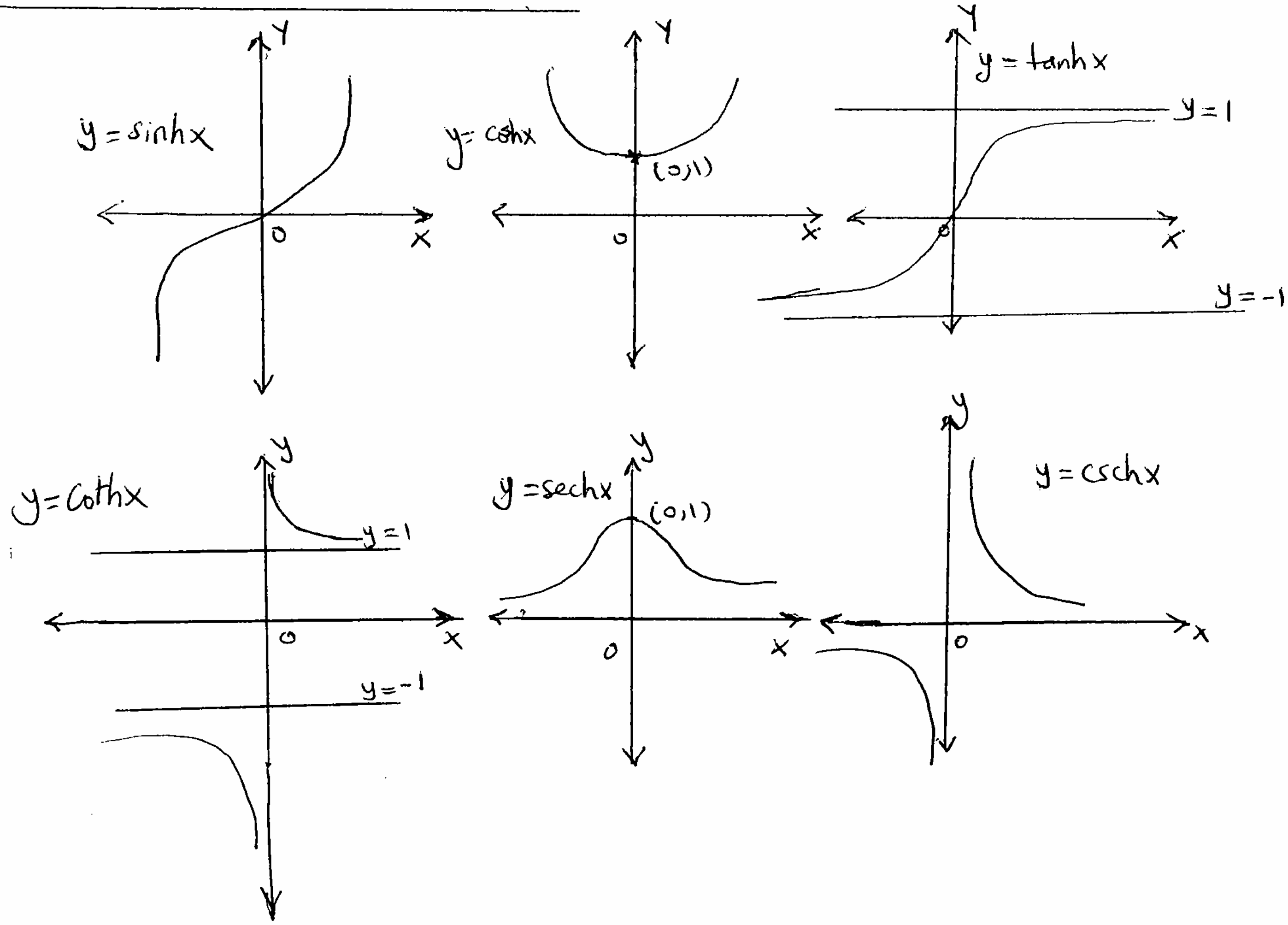
Defn. We define,

(1) $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (2) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(3) $\tanh = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (4) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{1}{\tanh x}$

(5) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$ (6) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Graph of Hyperbolic Functions



Some Important Relations And Identities بعض العلاقات والمتطابقات المهمة

(1) $\cosh^2 x - \sinh^2 x = 1$. (2) $\tanh^2 x + \text{sech}^2 x = 1$. (3) $\coth^2 x - \text{csch}^2 x = 1$.

(4) $\sinh(-x) = -\sinh x$ (5) $\cosh(-x) = \cosh x$. (6) $\tanh(-x) = -\tanh x$

(7) $\sinh x \pm \cosh x = \pm e^{\pm x}$.

(8) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.

(9) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

(10) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$.

(11) $\sinh 2x = 2 \sinh x \cosh x$.

(12) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$. (13) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$.

proof(1): $\cosh^2 x - \sinh^2 x = \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2$

$$= \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} = \frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\cancel{e^{2x}}+2-\cancel{e^{-2x}}}{4} = \frac{4}{4} = 1$$

Proof(2)

$$\tanh^2 x + \operatorname{sech}^2 x = \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x} = \frac{\sinh^2 x + 1}{\cosh^2 x} = \frac{\cosh^2 x}{\cosh^2 x} = 1 .$$

proof(4) $\sinh(-x) = \frac{e^{-x}-e^{-(-x)}}{2} = -\frac{e^{x}-e^{-x}}{2} = -\sinh x .$

proof(5) $\cosh(-x) = \frac{e^{-x}+e^{-(-x)}}{2} = \frac{e^{x}+e^{-x}}{2} = \cosh x .$

proof(6) $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x .$

proof(7) $\sinh x + \cosh x = \frac{e^{x}-e^{-x}}{2} + \frac{e^{x}+e^{-x}}{2} = \frac{e^{x}-e^{-x}+e^{x}+e^{-x}}{2}$

$$= \frac{2e^{x}}{2} = e^{x} .$$

EX.(1) If $\sinh x = -\frac{3}{4}$. find the value of the other hyperbolics

Solu.

$$\cosh^2 x = 1 + \sinh^2 x = 1 + \left(\frac{-3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow$$

Since $\cosh x > 0 \Rightarrow \cosh x = \frac{5}{4}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-\frac{3}{4}}{\frac{5}{4}} = \frac{-3}{4}$. and so on for the other hyperbolics.

H.W
EX.(2) Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

H.W
EX.(3) If $\sinh x = \tan\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Find the other hyperbolics in terms of the trigonometric.

H.W
EX.(4) Show that: $\sinh 3x = 3\sinh x + 4\sinh^3 x$.