

Applications of derivatives

Maxima and Minima :

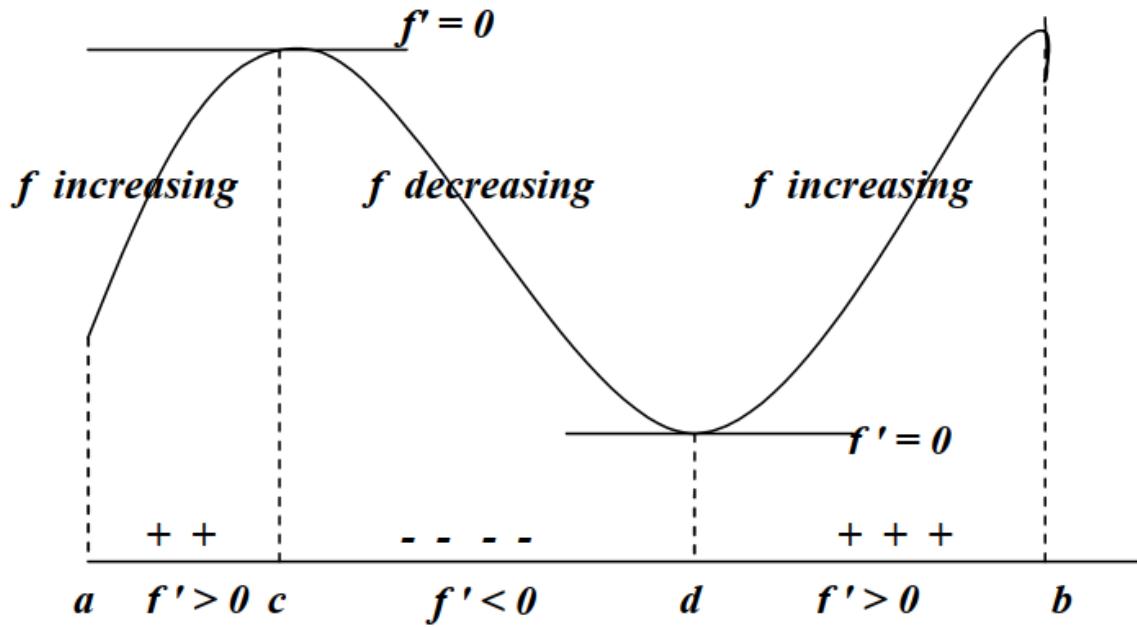
Increasing and decreasing function : Let f be defined on an interval and x_1, x_2 denoted a number on that interval :

- If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .
- If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .
- If $f(x_1) = f(x_2)$ for all values of x_1, x_2 then f is constant on that interval .

The first derivative test for rise and fall : Suppose that a function f has a derivative at every point x of an interval I . Then :

- f increases on I if $f'(x) > 0, \forall x \in I$
- f decreases on I if $f'(x) < 0, \forall x \in I$

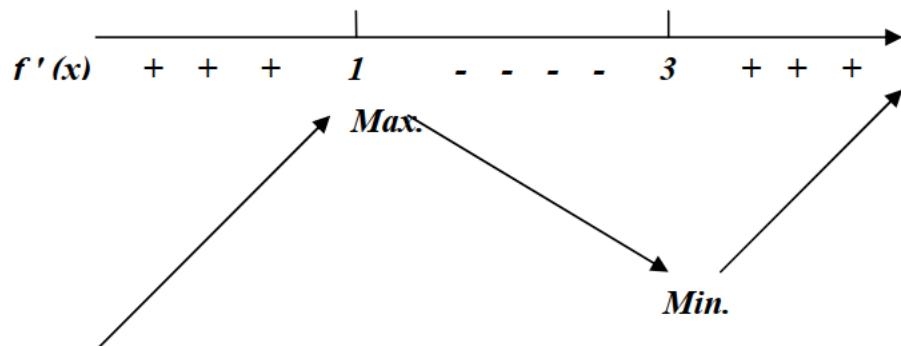
If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly , if f' changes from negative to positive values as x passes left to right through a point d , then the value of f at d is a local minimum value of f . That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d .

EX-5 – Graph the function : $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$.

$$\underline{Sol.} - f'(x) = x^2 - 4x + 3 \Rightarrow (x - 1)(x - 3) = 0 \Rightarrow x = 1, 3$$

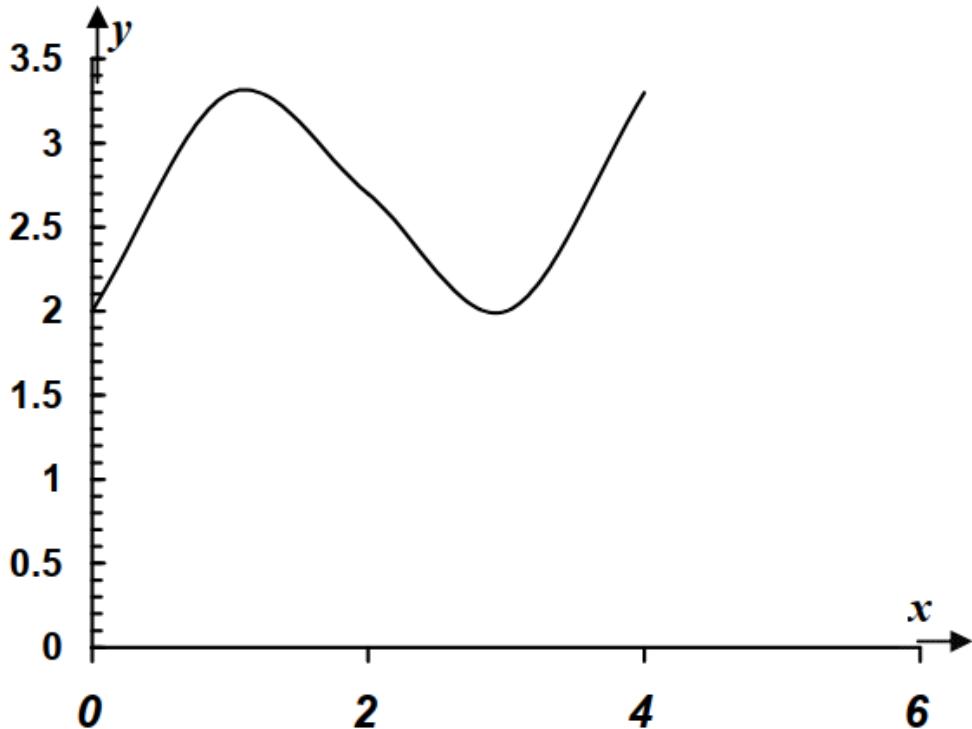


The function has a local maximum at $x = 1$ and a local minimum at $x = 3$.

To get a more accurate curve , we take :

x	0	1	2	3	4
$f(x)$	2	3.3	2.7	2	3.3

Then the graph of the function is :



Concave down and concave up : The graph of a differentiable function $y = f(x)$ is concave down on an interval where f' decreases , and concave up on an interval where f' increases.

The second derivative test for concavity : The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.

Point of inflection : A point on the curve where the concavity changes is called a point of inflection . Thus , a point of inflection on a twice – differentiable curve is a point where y'' is positive on one side and negative on other , i.e. $y''=0$.

EX-6 – Sketch the curve : $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

Sol. –

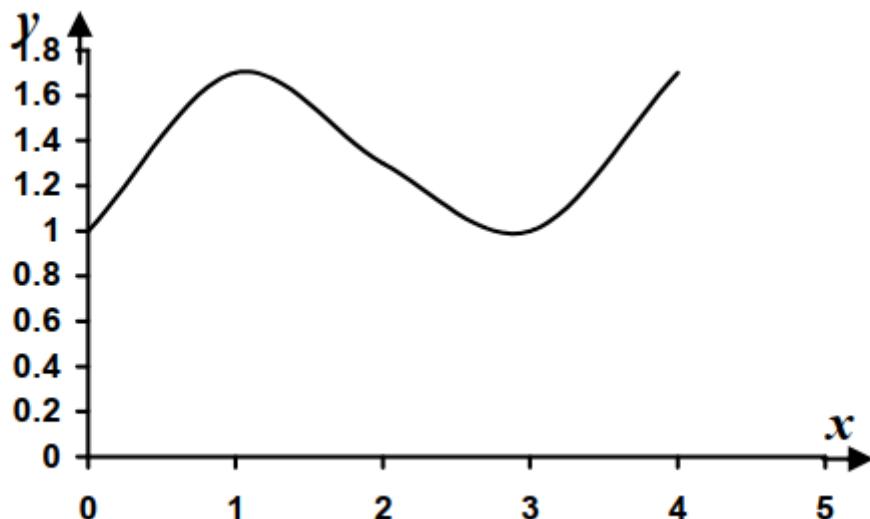
$$y' = \frac{1}{2}x^2 - 2x + \frac{3}{2} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3$$

$y'' = x - 2 \Rightarrow$ at $x = 1 \Rightarrow y'' = 1 - 2 = -1 < 0$ concave down .

\Rightarrow at $x = 3 \Rightarrow y'' = 3 - 2 > 0$ concave up .

\Rightarrow at $y'' = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$ point of inflection .

x	0	1	2	3	4
y	1	1.7	1.3	1	1.7



EX-7 – What value of a makes the function :

$$f(x) = x^2 + \frac{a}{x}, \text{ have :}$$

- i) a local minimum at $x = 2$?
- ii) a local minimum at $x = -3$?
- iii) a point of inflection at $x = 1$?
- iv) show that the function can't have a local maximum for any value of a .

Sol. –

$$f(x) = x^2 + \frac{a}{x} \Rightarrow \frac{df}{dx} = 2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3 \text{ and } \frac{d^2y}{dx^2} = 2 + \frac{2a}{x^3}$$

- i) at $x = 2 \Rightarrow a = 2 * 8 = 16$ and $\frac{d^2 f}{dx^2} = 2 + \frac{2 * 16}{2^3} = 6 > 0$ Mini.
- ii) at $x = -3 \Rightarrow a = 2(-3)^3 = -54$ and $\frac{d^2 f}{dx^2} = 2 + \frac{2(-54)}{(-3)^3} = 6 > 0$ Mini.
- iii) at $x = 1 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2a}{1} = 0 \Rightarrow a = -1$
- iv) $a = 2x^3 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2(2x^3)}{x^3} = 6 > 0$
Since $\frac{d^2 f}{dx^2} > 0$ for all value of x in $a = 2x^3$.
Hence the function don't have a local maximum .

H.W

. Find any local maximum and local minimum values , then sketch each curve by using first derivative :

- 1) $f(x) = x^3 - 4x^2 + 4x + 5$ (ans.: max.(0.7,6.2); min.(2,5))
- 2) $f(x) = \frac{x^2 - 1}{x^2 + 1}$ (ans.: min.(0,-1))
- 3) $f(x) = x^5 - 5x - 6$ (ans.: max.(-1,-2); min.(1,-10))
- 4) $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$ (ans.: min.(0.25,-0.47))

. Find the interval of x -values on which the curve is concave up and concave down , then sketch the curve :

- 1) $f(x) = \frac{x^3}{3} + x^2 - 3x$ (ans.: up(-1,∞); down(-∞,-1))
- 2) $f(x) = x^2 - 5x + 6$ (ans.: up(-∞,∞))
- 3) $f(x) = x^3 - 2x^2 + 1$ (ans.: up($\frac{2}{3}, \infty$); down($-\infty, \frac{2}{3}$))
- 4) $f(x) = x^4 - 2x^2$ (ans.: up($-\infty, -\frac{1}{\sqrt{3}}$), ($\frac{1}{\sqrt{3}}, \infty$); down($-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$))

Integration ← back

①

Introduction

We shall discuss a problem of finding a function $y=f(x)$ a derivative is given by the equation:

$$\frac{dy}{dx} = f(x) \quad \text{--- (1)}$$

equation (1) usually called a differential equation.

We shall restrict our attention on differential equations that contain a single derivative.

A function $y=F(x)$ is called a solution of the differential equation (1) if $\frac{d}{dx} F(x) = f(x)$.

The function $F(x)$ is also called a solution of $f(x)$.

Solve equation means to find all the functions that are solutions $f(x)$.

Theorem (1) If $F(x)$ is ~~a~~ a solution of $f(x)$, then $F(x)+C$ is also a solution, where C is any constant.

Proof :

Since $F(x)$ is a solution of $f(x)$ then $\frac{d}{dx} F(x) = f(x)$

$$\frac{d}{dx} [F(x)+C] = \frac{d}{dx} F(x) + \frac{d}{dx} C = f(x) + 0 = f(x)$$

Hence $F(x)+C$ is a solution of $f(x)$.

From theorem(1), we may say that, if $f(x)$ is any solution of the equation $\frac{dy}{dx} = f(x)$ then all solutions are given by the ~~function~~ formula (2)

$$y = F(x) + C \quad \text{--- (2)}$$

In definite Integral :

The set of all ~~solutions~~ of $f(x)$ is called indefinite integral of f with respect to x and denoted by:

$$y = \int f(x) dx = F(x) + C \quad \text{--- (3)}$$

where the symbol \int is called an "integral sign", the function $f(x)$ is called the integrand of the integral, C is called the constant of integration, and dx tell us that the variable of integration is x .

Ex.(1) Solve the differential equation $\frac{dy}{dx} = 3x^2$

Since $\frac{d}{dx} x^3 = 3x^2$

Then $y = \int 3x^2 dx = x^3 + C$.

Ex.(2) Solve $\frac{dy}{dx} = \sin x \cos x$

Since $\frac{d}{dx} (\frac{1}{2} \sin^2 x) = \sin x \cos x$

Then $y = \int \sin x \cos x dx = \frac{1}{2} \sin^2 x + C$.

(3)

Some Integration formulas

If $u = u(x)$

$$(1) \int \frac{du}{dx} dx = u(x) + C$$

$$(2) \int a u(x) dx = a \int u(x) dx , \text{ where } a \text{ is constant.}$$

$$(3) \int \{u_1(x) + u_2(x) + \dots + u_n(x)\} dx = \int u_1(x) dx + \int u_2(x) dx + \dots + \int u_n(x) dx$$

$$(4) \int u^n \frac{du}{dx} \cdot dx = \frac{u^{n+1}}{n+1} + C , \quad n \neq -1$$

$$\begin{aligned} \underline{\text{Ex. (3)}} \text{ Evaluate } I &= \int (3x^4 - 2x^3 + x^{\frac{1}{2}} + 2x^{-2} + 5x^{-\frac{1}{2}} - \sqrt{2}) dx \\ &= 3 \int x^4 dx - 2 \int x^3 dx + \int x^{\frac{1}{2}} dx + 2 \int x^{-2} dx + 5 \int x^{-\frac{1}{2}} dx - \sqrt{2} \int dx \\ &= \frac{3}{5} x^5 - \frac{1}{2} x^4 + \frac{2}{3} x^{\frac{3}{2}} - 2 x^{-1} + 10 x^{\frac{1}{2}} - \sqrt{2} x + C. \end{aligned}$$

$$\underline{\text{Ex. (4)}} \text{ Evaluate } I = \int \sqrt{(3x-1)^3}$$

$$\begin{aligned} I &= \int (3x-1)^{\frac{3}{2}} dx = \frac{1}{3} \int (3x-1)^{\frac{3}{2}} 3 dx = \frac{1}{3} \cdot \frac{2}{5} (3x-1)^{\frac{5}{2}} + C \\ &= \frac{2}{15} (3x-1)^{\frac{5}{2}} + C. \end{aligned}$$

$$\underline{\text{H.W}}$$

$$\underline{\text{Ex. (5)}} \int (x^2+1)^3 2x dx .$$

Definite Integrals

(4)

The integral $\int_a^b f(x) dx$ is called the definite integral of $f(x)$ over the interval $[a, b]$.

Later, we shall show that this integral is a number ~~defined~~ defined in a certain way as a limit of approximating sums over the interval from a to b on the x -axis.

Properties of definite Integrals

If $f(x)$ is a continuous function on $[a, b]$, then

$$\textcircled{1} \quad \int_a^b f(x) dx = - \int_b^a f(x) dx .$$

$$\textcircled{2} \quad \int_a^a f(x) dx = 0$$

$$\textcircled{3} \quad \int_a^b kf(x) dx = k \int_a^b f(x) dx , \quad k \text{ is constant.}$$

$$\textcircled{4} \quad \int_a^b \{ f_1(x) + f_2(x) + \dots + f_n(x) \} dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx + \dots + \int_a^b f_n(x) dx$$

$$\textcircled{5} \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad \text{for any } c \in [a, b].$$

"The fundamental Theorem of Integral Calculus"

If $f(x)$ is continuous function on $[a, b]$ and $F(x)$ is any solution of $f(x)$ over $[a, b]$, then

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a) .$$

(5)

Ex.(6) Evaluate $I = \int_{-3}^2 (6-x-x^2) dx$

$$I = \left(6x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_{-3}^2 = \left(12 - 2 - \frac{8}{3} \right) - \left(-18 - \frac{9}{2} + 9 \right) = \frac{125}{6}.$$

Ex.(7) Find $I = \int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = -(\cos \pi - \cos 0) = -(-1 - 1) = 2.$

Ans

Ex.(8) Find $I = \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\cos^2 2x} dx.$

Ex.(9) If $f(x)$ is a continuous, show that

$$\int_0^1 f(x) dx = \int_0^1 f(1-t) dt.$$

Proof

$$\text{Let } x = 1-t \Rightarrow dx = -dt$$

$$\text{at } x=0 \Rightarrow t=1$$

$$\text{at } x=1 \Rightarrow t=0$$

then

$$\int_0^1 f(x) dx = \int_1^0 f(1-t) dt = - \int_1^0 f(1-t) dt = \int_0^1 f(1-t) dt.$$

Methods of Integration

Integral Formulae (Standard Forms)

$$\textcircled{1} \int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1. \quad \textcircled{2} \int \frac{du}{u} = \ln u + C.$$

$$\textcircled{3} \int e^u du = e^u + C, \quad e = 2.7183. \quad \textcircled{4} \int a^u du = \frac{a^u}{\ln a} + C. \quad a > 0$$

$$\textcircled{5} \int \sin u du = -\cos u + C.$$

$$\textcircled{6} \int \cos u du = \sin u + C.$$

$$\textcircled{7} \int \sec^2 u du = \tan u + C.$$

$$\textcircled{8} \int \csc^2 u du = -\cot u + C.$$

$$\textcircled{9} \int \sec u \tan u du = \sec u + C$$

$$\textcircled{10} \int \csc u \cot u du = -\csc u + C$$

$$\textcircled{11} \int \sinh u du = \cosh u + C.$$

$$\textcircled{12} \int \cosh u du = \sinh u + C.$$

$$\textcircled{13} \int \operatorname{sech}^2 u du = \tanh u + C.$$

$$\textcircled{14} \int \operatorname{csch}^2 u du = -\coth u + C.$$

$$\textcircled{15} \int \operatorname{sech} u \tanh u du = -\operatorname{sech} u + C.$$

$$\textcircled{16} \int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C.$$

$$\textcircled{17} \int \frac{du}{\sqrt{a^2+u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$\textcircled{18} \int \frac{du}{a^2+u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$\textcircled{19} \int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$

$$\textcircled{20} \int \frac{du}{\sqrt{u^2-a^2}} = \sinh^{-1} \frac{u}{a} + C$$

$$\textcircled{21} \int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a} \tanh^{-1} \frac{u}{a} + C, & |u| < a \\ \frac{1}{a} \coth^{-1} \frac{u}{a} + C, & |u| > a \end{cases}$$

$$\textcircled{22} \int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1} \frac{u}{a} + C$$

$$\textcircled{23} \int \frac{du}{u\sqrt{u^2-a^2}} = -\frac{1}{a} \operatorname{sech}^{-1} \frac{u}{a} + C$$

$$\textcircled{24} \int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a} \operatorname{csch}^{-1} \frac{u}{a} + C.$$

Method [1] Integration By Substitution

(7)

The goal of this method is to transform the integral into a standard ~~form~~ form, to evaluate the integral

$$I = \int f[g(x)] g'(x) dx$$

Carry out the following steps:

1. Substitute $u = g(x)$ then $du = g'(x)dx$ to obtain $I = \int f(u) du$
2. Evaluate $I = \int f(u) du$ by integrating with respect to u .
3. Replace u by $g(x)$ in the final result.

EX.(1) Evaluate $\int \frac{dx}{\sqrt[3]{1-2x}}$

Solu.

$$I = \int (1-2x)^{-\frac{1}{3}} dx. \text{ Let } u = 1-2x \Rightarrow du = -2 dx$$

$$\begin{aligned} I &= -\frac{1}{2} \int (1-2x)^{-\frac{1}{3}} (-2) dx = -\frac{1}{2} \int u^{-\frac{1}{3}} du = -\frac{1}{2} \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\ &= -\frac{3}{4} (1-2x)^{\frac{2}{3}} + C. \end{aligned}$$

EX.(2) Find $I = \int \sin^2 5x \cos 5x dx$

Solu. Let $u = \sin 5x \Rightarrow du = 5 \cos 5x dx$

$$I = \frac{1}{5} \int u^2 du = \frac{1}{5} \frac{u^3}{3} + C = \frac{1}{15} \sin^3 5x + C.$$

EX.(3) Find $I = \int x e^{x^2+1} dx = \frac{1}{2} \int e^{x^2+1} 2x dx$

$$u = x^2 + 1 \Rightarrow du = 2x dx$$

$$I = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2+1} + C.$$

Ex.(4) Find $I = \int \frac{3 \cosh 3x}{4 + \sinh 3x} dx$

solt.

$$u = 4 + \sinh 3x \Rightarrow du = 3 \cosh 3x dx$$

$$I = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln u + C = \frac{1}{3} \ln(4 + \sinh 3x) + C.$$

Ex.(5) Evaluate $I = \int \frac{3 \cosh 3x}{4 + \sinh^2 3x} dx$

solt.

$$u = \sinh 3x \Rightarrow du = 3 \cosh 3x dx$$

$$I = \frac{1}{3} \int \frac{du}{4 + u^2} = \frac{1}{3} \frac{1}{2} \tan^{-1} \frac{u}{2} + C = \frac{1}{6} \tan^{-1} \left[\frac{\sinh 3x}{2} \right] + C.$$

Ex.(6) $I = \int \frac{dx}{1 + e^x} = \int \frac{dx}{1 + \frac{1}{e^{-x}}} = - \int \frac{-e^x dx}{e^x + 1} = -\ln(e^x + 1) + C.$

Ex.(7) Find $I = \int \sec x dx = \int \sec x \cdot \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$$\begin{aligned} u &= \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x) dx \\ &= \sec x (\tan x + \sec x) dx \end{aligned}$$

$$I = \int \frac{du}{u} = \ln u + C = \ln(\sec x + \tan x) + C.$$

Ex.(8) Evaluate $I = \int_0^{\sqrt{2}} \frac{x dx}{\sqrt{1-x^4}} = \frac{1}{2} \int_0^{\sqrt{2}} \frac{2x dx}{\sqrt{1-(x^2)^2}}$

$$u = x^2 \Rightarrow du = 2x dx . \text{ at } x=0 \Rightarrow u=0$$

$$\text{at } x=\frac{1}{\sqrt{2}} \Rightarrow u=\frac{1}{2}$$

$$I = \frac{1}{2} \int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u \Big|_0^{\frac{1}{2}} = \frac{1}{2} \left[\sin^{-1} \frac{1}{2} - \sin^{-1} 0 \right] = \frac{1}{2} \left(\frac{\pi}{6} - 0 \right) = \frac{\pi}{12}.$$

Exercises To Solve [No. 1]

(9)

$$\textcircled{1} \int \left(x - \frac{1}{x}\right)^2 dx$$

$$\textcircled{2} \int \frac{1 + e^{2x}}{e^x} dx$$

$$\textcircled{3} \int \frac{\sec^2 x dx}{1 + \tan^2 x}$$

$$\textcircled{4} \int \frac{e^x}{1 + e^{2x}} dx$$

$$\textcircled{5} \int \frac{dx}{x[1 + (\ln x)^2]}$$

$$\textcircled{6} \int \tan^2 3x dx$$

$$\textcircled{7} \int \operatorname{sech} x dx$$

$$\textcircled{8} \int \frac{\sec^2(\ln x) dx}{x}$$

$$\textcircled{9} \int \tan x \ln(\cos x) dx$$

$$\textcircled{10} \int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx$$

$$\textcircled{11} \int_0^1 \frac{\sqrt{1 + e^{-2x}}}{e^{-3x}} dx$$

$$\textcircled{12} \int_4^9 \frac{dx}{x - \sqrt{x}}$$

$$\textcircled{13} \int_0^1 \frac{(1 + e^{-2x})^{\frac{1}{2}}}{e^{-3x}} dx$$

$$\textcircled{14} \int_6^\infty \frac{dx}{e^{-x} - e^x}$$

$$\textcircled{15} \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

$$\textcircled{16} \int \tan 4x dx$$

$$\textcircled{17} \int \frac{x-2}{\sqrt{9-x^2}} dx$$

$$\textcircled{18} \int \frac{x\sqrt{x}}{1+x^5} dx$$

$$\textcircled{19} \int \operatorname{csch} x dx$$

$$\textcircled{20} \int \csc x dx$$

Method [2]

Certain Powers of Trigonometric And Hyperbolic Integral

Consider the following integrals forms :

- (A) $\int \sin^m u \cos^n u du$ or $\int \sinh^m u \cosh^n u du$
- (B) $\int \tan^m u \sec^n u du$ or $\int \tanh^m u \operatorname{sech}^n u du$
- (C) $\int \cot^m u \csc^n u du$ or $\int \coth^m u \operatorname{csch}^n u du$.

Under Type(A), there are three cases :

Case I If m is odd and +ive , we factor out $\sin u$ ($\sinh u$) and change the remaining even power of $\sin u$ ($\sinh u$) to $\cos u$ ($\cosh u$) using the identities:

$$\sin^2 u = 1 - \cos^2 u \quad , \quad \sinh^2 u = \cosh^2 u - 1 .$$

Ex.(1) Find $I = \int \sin^5 2x \cos^{-\frac{3}{2}} 2x dx = \int \sin^4 2x \sin 2x \cos^{-\frac{3}{2}} 2x dx$

$$= \int (1 - \cos^2 2x)^2 \sin 2x \cos^{-\frac{3}{2}} 2x dx = \int (1 - 2\cos^2 2x + \cos^4 2x) \cdot \sin^2 2x \cdot \cos^{-\frac{3}{2}} 2x dx$$

$$= \int (\cos^{-\frac{3}{2}} 2x - 2\cos^{-\frac{1}{2}} 2x + \cos^{-\frac{5}{2}} 2x) \sin 2x dx$$

$$= -\frac{1}{2} \left[\frac{\cos^{-\frac{1}{2}} 2x}{-\frac{1}{2}} - 2 \frac{\cos^{-\frac{3}{2}} 2x}{\frac{1}{2}} + \frac{\cos^{-\frac{7}{2}} 2x}{\frac{7}{2}} \right] + C = \cos^{-\frac{1}{2}} 2x + \frac{2}{3} \cos^{-\frac{3}{2}} 2x - \frac{2}{7} \cos^{-\frac{5}{2}} 2x + C .$$

Case II

If n is odd and +ive , we factor out $\cos u$ ($\cosh u$) and change the remaining even power of $\cos u$ ($\cosh u$) to $\sin u$ ($\sinh u$) using the identities :-

$$\cos^2 u = 1 - \sin^2 u \quad , \quad \cosh^2 u = 1 + \sinh^2 u .$$

$$\underline{\text{Ex.}(2)} \text{ find } I = \int \sinh^4 3x \cosh^3 3x dx = \int \cosh^2 3x \sinh^4 3x \cosh 3x dx$$

$$I = \int (1 + \sinh^2 3x) \sinh^4 3x \cosh 3x dx = \int (\sinh^6 3x + \sinh^4 3x) \cosh 3x dx \\ = \frac{1}{3} \left[\frac{\sinh^7 3x}{7} + \frac{\sinh^5 3x}{5} \right] + C = \frac{1}{21} \sinh^7 3x + \frac{1}{15} \sinh^5 3x + C .$$

Case III

If both m and n are even and +ive (or one of them zero) we reduce the degree of the expression by using the identities !-

$$\sin^2 u = \frac{1 - \cos 2u}{2} \quad , \quad \sinh^2 u = \frac{\cosh 2u - 1}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2} \quad , \quad \cosh^2 u = \frac{\cosh 2u + 1}{2}$$

$$\underline{\text{Ex.}(3)} \quad I = \int \sin^2 2x \cos^2 2x dx = \frac{1}{4} \int (1 - \cos 4x)(1 + \cos 4x) dx$$

$$= \frac{1}{4} \int (1 - \cos^2 4x) dx = \frac{1}{4} \int \left(1 - \frac{1 + \cos 8x}{2}\right) dx = \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 8x\right) dx$$

$$= \frac{1}{4} \left[\frac{x}{2} - \frac{1}{16} \sin 8x \right] + C .$$

Under Type (B), there are two cases

Case I If n is even and +ive , we factor out $\sec^2 u$ ($\operatorname{sech}^2 u$) and change the remaining even power of $\sec u$ ($\operatorname{sech} u$) to $\tan u$ ($\tanh u$) using the identities : $\sec^2 u = 1 + \tan^2 u \quad , \quad \operatorname{sech}^2 u = 1 - \tanh^2 u .$

(12)

Ex.(4) Find $I = \int \operatorname{sech}^4 \frac{x}{2} \tanh^{-\frac{1}{3}} \frac{x}{2} dx$

$$= \int \operatorname{sech}^2 \frac{x}{2} \tanh^{-\frac{1}{3}} \frac{x}{2} \operatorname{sech}^2 \frac{x}{2} dx$$

$$= \int (1 - \tanh^2 \frac{x}{2}) \tanh^{-\frac{1}{3}} \frac{x}{2} \operatorname{sech}^2 \frac{x}{2} dx = \int (\tanh^{-\frac{1}{3}} \frac{x}{2} - \tanh^{\frac{5}{3}} \frac{x}{2}) \operatorname{sech}^2 \frac{x}{2} dx$$

$$= 2 \left[\frac{\tanh^{\frac{2}{3}} \frac{x}{2}}{\frac{2}{3}} - \frac{\tanh^{\frac{8}{3}} \frac{x}{2}}{\frac{8}{3}} \right] + C = \frac{1}{3} \tanh^{\frac{2}{3}} \frac{x}{2} - \frac{3}{4} \tanh^{\frac{8}{3}} \frac{x}{2} + C$$

Case II If m is odd and +ive, we factor out $\sec u \tan u$ ($\operatorname{sech} u \operatorname{tanh} u$) and change remaining even power of $\tan u$ ($\operatorname{tanh} u$) to $\operatorname{sech} u$ ($\operatorname{sech} u$) using the identities :-

$$\tan^2 u = \sec^2 u - 1 \quad , \quad \tanh^2 u = 1 - \operatorname{sech}^2 u .$$

Ex.(5) $I = \int \tan^3 2x \sec^{-\frac{1}{4}} 2x dx = \int (\tan^2 2x \sec^{-\frac{5}{4}} 2x) \sec 2x \tan 2x dx$

$$= \int (\sec^2 2x - 1) \sec^{-\frac{5}{4}} 2x \sec 2x \tan 2x dx = \int (\sec^{\frac{3}{4}} 2x - \sec^{-\frac{5}{4}} 2x) \sec 2x \tan 2x dx$$

$$= \frac{1}{2} \left[\frac{\sec^{\frac{7}{4}} 2x}{\frac{7}{4}} - \frac{\sec^{-\frac{1}{4}} 2x}{-\frac{1}{4}} \right] + C = \frac{2}{7} \sec^{\frac{7}{4}} 2x + 2 \sec^{-\frac{1}{4}} 2x + C .$$

Under Type(C), there are two cases similar to those of type (B) where the identities :-

$$\csc^2 u = \cot^2 u + 1 \quad , \quad \operatorname{csch}^2 u = \coth^2 u - 1 .$$

Exercises To Solve [No. 2]

$$\textcircled{1} \int \sin^5 2x \, dx$$

$$\textcircled{2} \int \cot^4 3x \, dx$$

$$\textcircled{3} \int \cot^3 2x \csc^4 2x \, dx$$

$$\textcircled{4} \int_0^{\pi} \cos^2 x \, dx$$

$$\textcircled{5} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3 2x \, dx$$

$$\textcircled{6} \int_0^1 \sinh^4 x \, dx$$

$$\textcircled{7} \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$$

$$\textcircled{8} \int \sin^4 x \cos^2 x \, dx$$

$$\textcircled{9} \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$$

$$\textcircled{10} \int x \sin^3 x^2 \, dx . \quad \textcircled{11} \int \cos^3 x \sin^{-\frac{1}{2}} x \, dx$$

$$\textcircled{12} \int \sin^6 x \, dx$$

$$\textcircled{13} \int \csc^6 x \, dx$$

$$\textcircled{14} \int_0^{\frac{\pi}{3}} \tan^3 x \sec x \, dx$$

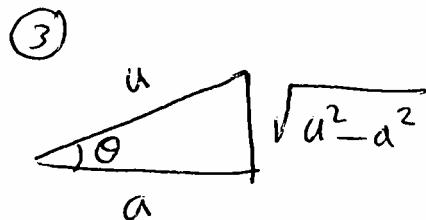
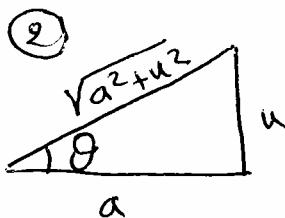
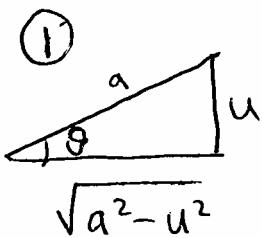
$$\textcircled{15} \int \tan \frac{x}{3} \sec^3 \frac{x}{3} \, dx .$$

Method [3] Trigonometric Substitutions

(14)

If the integral involve one of the forms a^2+u^2 , $\sqrt{a^2-u^2}$, $\sqrt{u^2+a^2}$, or $\sqrt{u^2-a^2}$. Then the substitutions as follows:

- ① If $\sqrt{a^2-u^2}$, let $u=a\sin\theta \Rightarrow a^2-u^2=a^2\cos^2\theta$.
- ② If $\sqrt{a^2+u^2}$, let $u=a\tan\theta \Rightarrow a^2+u^2=a^2\sec^2\theta$.
- ③ If $\sqrt{u^2-a^2}$, let $u=a\sec\theta \Rightarrow u^2-a^2=a^2\tan^2\theta$.



EX.(1) Find $I = \int \frac{dx}{4+x^2} = \frac{1}{2} \tan^{-1} \frac{x}{2} + C$.

or

$$x=2\tan\theta \Rightarrow \tan\theta = \frac{x}{2} \Rightarrow \theta = \tan^{-1} \frac{x}{2}$$

$$dx = 2\sec^2\theta d\theta$$

$$I = \int \frac{2\sec^2\theta d\theta}{4+4\tan^2\theta} = \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta} = \frac{1}{2} \int d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} + C.$$

EX.(2) Find $I = \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2} dx$

$$x = \sin\theta \quad \text{at } x = -\frac{1}{2} \Rightarrow -\frac{1}{2} = \sin\theta \Rightarrow \theta = -\frac{\pi}{6}$$

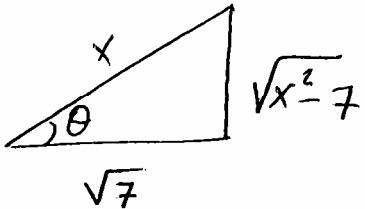
$$dx = \cos\theta d\theta \quad \text{at } x = \frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$$

$$\begin{aligned}
 I &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1 - \sin^2 \theta} \cos \theta d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 \theta d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1 + \cos 2\theta}{2} d\theta \\
 &= \frac{1}{2} \left[(\theta + \frac{1}{2} \sin 2\theta) \right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2} \left[\left(\frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} \right) - \left(-\frac{\pi}{6} + \frac{1}{2} \sin \left(-\frac{\pi}{3} \right) \right) \right] \\
 &= \frac{1}{2} \left[\frac{\pi}{3} + \frac{1}{2} \frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{1}{2} \frac{\sqrt{3}}{2} \right] = \frac{1}{2} \left[\frac{\pi}{2} + \frac{\sqrt{3}}{2} \right] = \frac{\pi + \sqrt{3}}{4}.
 \end{aligned}$$

Ex.(3) Find

$$I = \int \frac{\sqrt{x^2 - 7}}{x} dx$$

$$x = \sqrt{7} \sec \theta \Rightarrow \sec \theta = \frac{x}{\sqrt{7}}$$



$$\Rightarrow \theta = \sec^{-1} \frac{x}{\sqrt{7}}$$

$$dx = \sqrt{7} \sec \theta \tan \theta d\theta.$$

$$I = \int \frac{\sqrt{7 \sec^2 \theta - 7}}{\sqrt{7} \sec \theta} \cdot \sqrt{7} \sec \theta \tan \theta d\theta = \int \sqrt{7} \tan^2 \theta d\theta$$

$$= \sqrt{7} \int (\sec^2 \theta - 1) d\theta = \sqrt{7} [\tan \theta - \theta] + C = \sqrt{7} \left[\tan \left[\sec^{-1} \frac{x}{\sqrt{7}} \right] - \sec^{-1} \frac{x}{\sqrt{7}} \right] + C$$

$$+ C.$$

$$= \sqrt{7} \left[\frac{1}{\sqrt{7}} \sqrt{x^2 - 7} - \sec^{-1} \frac{x}{\sqrt{7}} \right] + C.$$

H.W
Ex.(4)

Evaluate $I = \int x^3 \sqrt{9+x^2} dx.$

Exercises To Solve [No. 3]

(16)

$$\textcircled{1} \int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$$

$$\textcircled{2} \int_0^2 \frac{x^2 dx}{x^2+4}$$

$$\textcircled{3} \int_0^{2\sqrt{3}} \frac{x^3 dx}{\sqrt{x^2+4}}$$

$$\textcircled{4} \int_0^1 \frac{x^2 dx}{(4-x^2)^{3/2}}$$

$$\textcircled{5} \int \frac{\sin \theta}{\sqrt{2-\cos^2 \theta}} d\theta$$

$$\textcircled{6} \int \frac{dx}{(9-x^2)^{3/2}}$$

$$\textcircled{7} \int_0^{\sqrt{5}} x^2 \sqrt{5-x^2} dx$$

$$\textcircled{8} \int_1^3 \frac{dx}{x^4 \sqrt{x^2+3}}$$

$$\textcircled{9} \int \frac{dx}{x^4 \sqrt{4-x^2}}$$

$$\textcircled{10} \int \frac{\sqrt{4x^2-9}}{x} dx .$$

Method [4] Hyperbolic Substitutions

As in method [3], the hyperbolic substitutions can be used to the forms a^2+u^2 , $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, or $\sqrt{u^2-a^2}$.

(1) If $\sqrt{a^2-u^2}$, let $u=a \tanh v$ or $u=a \operatorname{sech} v$.

(2) If $\sqrt{a^2+u^2}$, let $u=a \sinh v$ or $u=a \operatorname{csch} v$.

(3) If $\sqrt{u^2-a^2}$, let $u=a \cosh v$ or $u=a \operatorname{coth} v$.

Ex.(1) Find $I = \int \sec x dx$ [see method [1], Ex.(7)]

$$u = \sec x \Rightarrow du = \sec x \tan x dx \Rightarrow du = \sec x \sqrt{\sec^2 x - 1} dx$$

$$I = \int u \cdot \frac{du}{u \sqrt{u^2-1}} = \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C = \ln(u + \sqrt{u^2-1}) + C$$

$$= \ln(\sec x + \sqrt{\sec^2 x - 1}) + C$$

$$= \ln(\sec x + \tan x) + C .$$

EX.(2) Evaluate $I = \int \frac{dx}{x^3 \sqrt{x^2+4}}$

Solu.

$$x = 2 \sinh v \Rightarrow dx = 2 \cosh v dv$$

$$I = \int \frac{2 \cosh v dv}{8 \sinh^3 v \cdot 2 \cosh v} = \int \frac{dv}{8 \sinh^3 v} = \frac{1}{8} \int \operatorname{csch}^3 v dv$$

is not easy to find.

We try $x = 2 \operatorname{csch} v \Rightarrow v = \operatorname{csch}^{-1} \frac{x}{2}$

$$dx = -2 \operatorname{csch} v \coth v dv$$

$$I = \int \frac{-2 \operatorname{csch} v \coth v dv}{8 \operatorname{csch}^3 v \cdot 2 \coth v} = -\frac{1}{8} \int \sinh^2 v dv = -\frac{1}{8} \int \left(\frac{\cosh 2v - 1}{2} \right) dv$$

$$= -\frac{1}{16} \left[\frac{1}{2} \sinh 2v - v \right] + C = \frac{1}{16} [\sinh v \cosh v - v] + C$$

$$= -\frac{1}{16} \left[\frac{2}{x} \cdot \frac{\sqrt{x^2+4}}{x} - \operatorname{csch}^{-1} \frac{x}{2} \right] + C.$$

EX.(3) Find $I = \int \frac{x^3 dx}{(4-x^2)^{3/2}}$

Solu.

$$x = 2 \tanh v \Rightarrow dx = 2 \operatorname{sech}^2 v dv$$

$$I = \int \frac{8 \tanh^3 v \cdot 2 \operatorname{sech}^2 v dv}{(4 - 4 \tanh^2 v)^{3/2}} = \int \frac{16 \tanh^3 v \operatorname{sech}^2 v dv}{(4 \operatorname{sech}^2 v)^{3/2}}$$

$$= \int \frac{16 \tanh^3 v \operatorname{sech}^2 v dv}{8 \operatorname{sech}^3 v} = 2 \int \tanh^3 v (\operatorname{sech} v)^{-1} dv$$

$$= 2 \int (-\operatorname{sech}^2 v + 1) \operatorname{sech} v - \operatorname{sech} v \tanh v dv$$

$$= 2 \int (\operatorname{sech}^2 v - 1) \operatorname{sech} v \tanh v dv = 2 \left[-\frac{(\operatorname{sech} v)^2}{-1} + \operatorname{sech} v \right] + C$$

$$= 2 [\cosh v + \operatorname{sech} v] + C = 2 \left[\frac{2}{\sqrt{4-x^2}} + \frac{\sqrt{4-x^2}}{2} \right] + C.$$

Exercises To Solve [NO. 4]

(1) $\int_0^3 \frac{dx}{\sqrt{x^2+9}}$

(2) $\int_2^3 \sqrt{x^2-4} dx$

(3) $\int \frac{x^2 dx}{(9+x^2)^{1/2}}$

(4) $\int_2^4 \frac{\sqrt{x^2-4}}{x^2} dx$

(5) $\int (3+x^2)^{3/2} dx$

(6) $\int x^2 (5+x^2)^{1/2} dx$

(7) $\int \frac{x^3 dx}{(5-x^2)^{3/2}}$

(8) $\int x^2 \sqrt{5-x^2} dx$

(9) $\int \csc x dx$

(10) $\int \operatorname{sech} x dx$

(11) $\int \sec^3 x dx$

(12) $\int \csc^3 x dx$.

Method [5]Integrals Involving Quadratic functions

If the integral involve a quadratic function x^2+ax+b , we reduced it to the form u^2+B by completing the square as follows :-

$$x^2+ax+b = x^2+ax+\frac{a^2}{4} + b - \frac{a^2}{4} = \left(x+\frac{a}{2}\right)^2 + \left(b-\frac{a^2}{4}\right) = u^2 + B$$

where $u=x+\frac{a}{2}$ and $B=b-\frac{a^2}{4}$. Then the solution can be found by method [3] or [4].

EX.(1) Evaluate $I = \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-2x+1-1)}} = \int \frac{dx}{\sqrt{-[(x-1)^2-1]}}$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}} .$$

Let $u=x-1 \Rightarrow du=dx \Rightarrow I = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(x-1) + C.$

$$\underline{\text{EX.(2)}} \quad I = \int \frac{(4x+5) dx}{(x^2 - 2x + 2)} = \int \frac{(4x+5) dx}{(x^2 - 2x + 1 + 1)^{3/2}}$$

$$= \int \frac{(4x+5) dx}{[(x-1)^2 + 1]^{3/2}}$$

$$u = x-1 \Rightarrow du = dx$$

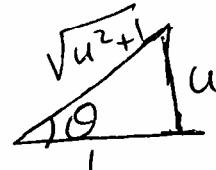
$$x = u+1$$

$$I = \int \frac{[4(u+1)+5] du}{(u^2+1)^{3/2}} = \int \frac{4u+9}{(u^2+1)^{3/2}} du = \int \frac{4u du}{(u^2+1)^{3/2}} + \int \frac{9 du}{(u^2+1)^{3/2}}$$

$$= 2 \int 2u(u^2+1)^{-3/2} du + 9 \int \frac{du}{(u^2+1)^{3/2}} = 2 \frac{(u^2+1)^{-1/2}}{-1/2} + 9 \int \frac{du}{(u^2+1)^{3/2}}$$

$$\text{Consider } I_1 = \int \frac{du}{(u^2+1)^{3/2}}$$

$$u = \tan \theta \Rightarrow du = \sec^2 \theta d\theta$$



$$I_1 = \int \frac{\sec^2 \theta d\theta}{[\sec^2 \theta]^{3/2}} = \int \cos \theta d\theta = \sin \theta + C = \frac{u}{\sqrt{u^2+1}} + C$$

$$\therefore I = -\frac{4}{\sqrt{u^2+1}} + \frac{9u}{\sqrt{u^2+1}} + C = \frac{9u-4}{\sqrt{u^2+1}} + C = \frac{9(x-1)-4}{\sqrt{(x-1)^2+1}} + C$$

$$= \frac{9x-13}{(x^2-2x+2)^{1/2}} + C.$$

Exercises To Solve [No. 5]

$$\textcircled{1} \int_1^2 \frac{dx}{x^2 + 2x + 5}$$

$$\textcircled{2} \int_1^2 \frac{3dx}{9x^2 - 6x + 5}$$

$$\textcircled{3} \int_{-1}^0 \frac{dx}{\sqrt{3-2x-x^2}}$$

$$\textcircled{4} \int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}$$

$$\textcircled{5} \lim_{a \rightarrow 5} \int_a^{-4} \frac{dx}{\sqrt{-x^2-8x-15}}$$

$$\textcircled{6} \int \frac{\cos x dx}{\sin^2 x + 2\sin x + 5}$$

Method [6]Integration By Parts

20

Consider $w = u \cdot v$

$$dw = u dv + v du \Rightarrow u dv = dw - v du$$

$$\int u dv = \int dw - \int v du = w - \int v du$$

$\boxed{\int u dv = w - \int v du}$

Ex.(1) Find $I = \int \ln x dx$

$$u = \ln x, \quad dv = dx \\ du = \frac{1}{x} dx, \quad v = x$$

$$I = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

Ex.(2) Find $I = \int \tan^{-1} x dx$

$$u = \tan^{-1} x, \quad dv = dx \\ du = \frac{dx}{1+x^2}, \quad v = x$$

$$I = x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C.$$

Ex.(3) Find $I = \int x \ln x dx = \int \ln x \cdot x dx$

$$u = \ln x, \quad dv = x dx \\ du = \frac{dx}{x}, \quad v = \frac{1}{2} x^2.$$

$$I = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C.$$

Ex.(4) Evaluate $I = \int x e^x dx$

$$u = x, \quad dv = e^x dx$$

$$du = dx, \quad v = e^x$$

$$I = x e^x - \int e^x dx = x e^x - e^x + C.$$

Ex.(5) Evaluate $I = \int \sin^3 4x dx$.

Ex-(6) Find $I = \int x^2 e^x dx$

$$u = x^2, dv = e^x dx$$

$$du = 2x dx, v = e^x$$

$$I = x^2 e^x - \int 2x e^x dx$$

$$\text{use Ex. (4), we have } \Rightarrow I = x^2 e^x - 2(x e^x - e^x + C) \\ = x^2 e^x - 2x e^x + 2e^x + K$$

$$\text{where } K = -2C.$$

Tabular Integration

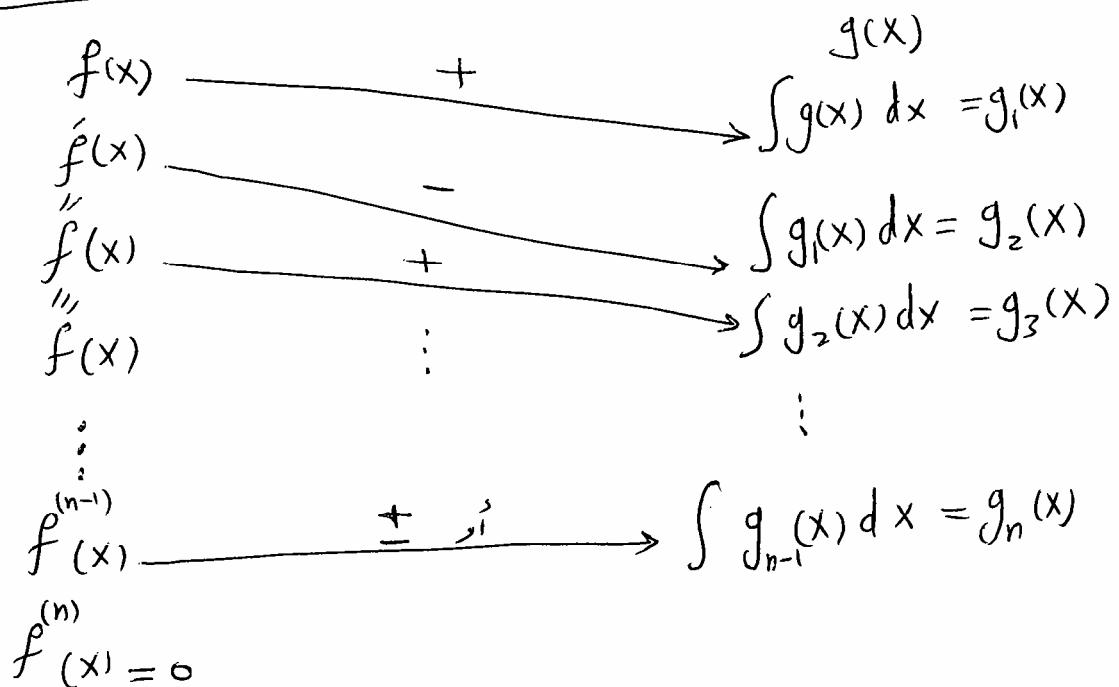
Consider the integral of the form : $\int f(x) g(x) dx$
 in which $f(x)$ can be differentiable repeatedly to become zero and $g(x)$ can be integrated repeatedly without difficulty.

Tabular integration saves a great deal of work as natural method for integration by parts.

The method can be illustrated as follows:-

$f(x)$ and its derivatives

$g(x)$ and its integrals

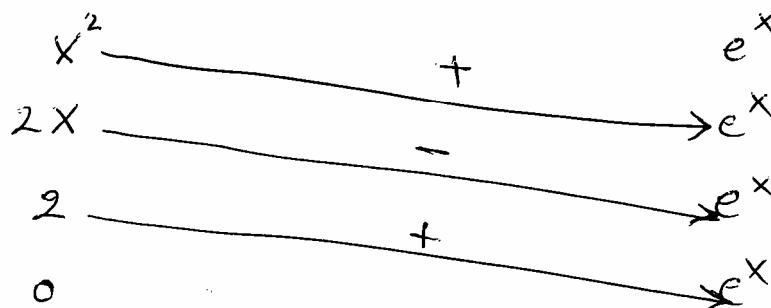


$$\int f(x)g(x)dx = f(x)g_1(x) - \bar{f}'(x)g_2(x) + \bar{\bar{f}}'(x)g_3(x) + \dots \pm \overset{(n-1)}{\bar{f}^{(n)}}(x)g_n(x) + C.$$

Ex.(7) Back to Ex.(6), we have $\int x^2 e^x dx$.

Let $f(x) = x^2$ and $g(x) = e^x$

$f(x)$ and its derivatives



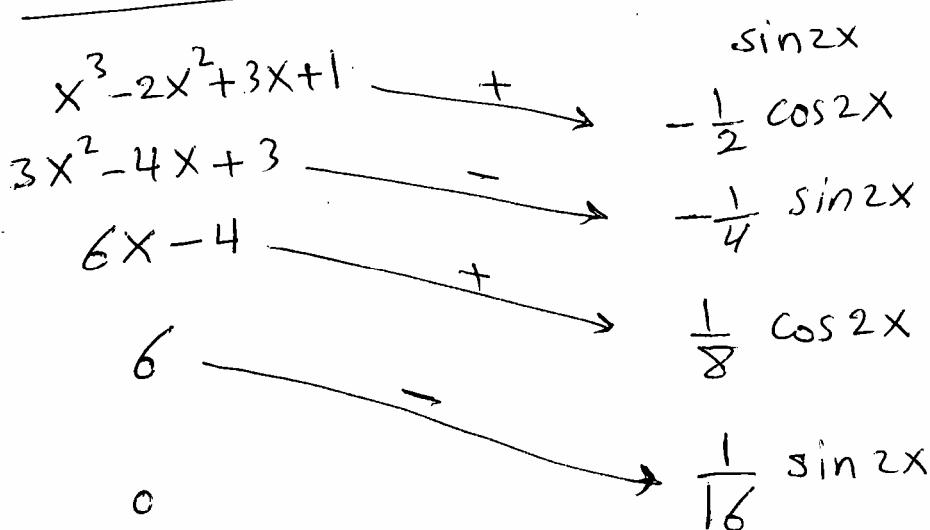
$g(x)$ and its integrals

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C.$$

Ex.(8) Evaluate $I = \int (x^3 - 2x^2 + 3x + 1) \sin 2x dx$

Let $f(x) = x^3 - 2x^2 + 3x + 1$, $g(x) = \sin 2x$

$f(x)$ and its derivatives



$g(x)$ and its integrals

$$I = \int (x^3 - 2x^2 + 3x + 1) \sin 2x dx = (x^3 - 2x^2 + 3x + 1) \left(-\frac{1}{2} \cos 2x\right) + \frac{1}{4} (3x^2 - 4x + 3) \sin 2x + \frac{1}{8} (6x - 4) \cos 2x - \frac{6}{16} \sin 2x.$$

Ex.(9) Evaluate $I = \int e^x \sin x dx$.

$$u = e^x, dv = \sin x \Rightarrow du = e^x dx, v = -\cos x$$

$$I = -e^x \cos x + \int e^x \cos x dx = -e^x \cos x + I_1$$

$$\text{where } I_1 = \int e^x \cos x dx$$

$$u = e^x, dv = \cos x dx$$

$$du = e^x dx, v = \sin x$$

$$I_1 = e^x \sin x - \int e^x \sin x dx = e^x \sin x - I$$

$$\therefore I = -e^x \cos x + e^x \sin x - I$$

$$2I = -e^x \cos x + e^x \sin x$$

$$I = \frac{e^x}{2} (\sin x - \cos x) + C.$$

Exercises To Solve [No. 6]

- ① $\int x^2 \ln(x+1) dx$
- ② $\int x^{-2} \tan^{-1} x dx$
- ③ $\int_0^1 x \sqrt{1-x} dx$
- ④ $\int_0^\infty x^3 e^{-x} dx$
- ⑤ $\int \sin(\ln x) dx$
- ⑥ $\int \sinh 2x \cosh 5x dx$
- ⑦ $\int e^{-x} \sin x dx$
- ⑧ $\int x [\ln(x)]^2 dx$
- ⑨ $\int \sin \sqrt{2x} dx$
- ⑩ $\int x^2 \tan^{-1} x dx$
- ⑪ $\int \ln(x + \sqrt{1+x^2}) dx$
- ⑫ $\int \sqrt{1-x^2} \sin^{-1} x dx$

Method [7] Integration of Rational Functions

(24)

Defn. A rational function is a quotient of two polynomials, written as

$$R(x) = \frac{P_n(x)}{Q_m(x)}, \quad Q_m(x) \neq 0 \text{ where } P_n(x) \text{ and } Q_m(x) \text{ are polynomials of degree } n \text{ and } m \text{ respectively.}$$

If $n > m$, we perform a long division until we obtain a rational function whose numerator degree less than or equal to the denominator degree

for Example $I = \int \frac{x^5 - 6x^4 - 2x^2 - 3x + 4}{x^3 + 2x + 3} dx$

$$\begin{array}{r} x^2 - 6x - 2 \\ \hline x^3 + 2x + 3 \end{array} \overline{\left[\begin{array}{r} x^5 - 6x^4 - 2x^2 - 3x + 4 \\ - x^5 - 2x^3 - 3x^2 \\ \hline - 6x^4 - 2x^3 - 5x^2 - 3x + 4 \\ - 6x^4 - 12x^3 - 18x^2 \\ \hline - 2x^3 + 7x^2 + 15x + 4 \\ - 2x^3 - 4x^2 - 6 \\ \hline 7x^2 + 19x + 10 \end{array} \right]}$$

$$\begin{aligned} \therefore I &= \int \left(x^2 - 6x - 2 + \frac{7x^2 + 19x + 10}{x^3 + 2x + 3} \right) dx \\ &= \frac{1}{3}x^3 - 3x^2 - 2x + \int \frac{7x^2 + 19x + 10}{x^3 + 2x + 3} dx. \end{aligned}$$

If $n \leq m$, we shall discuss the three cases of separating $\frac{P_n(x)}{Q_m(x)}$ in a sum of partial fractions.

Case I If the m factors of $Q_m(x)$ are all different and simple, that is,

$Q_m(x) = (x-a_1)(x-a_2) \cdots (x-a_m)$. Then we assign the sum of m partial fractions to these factors as follows

$$\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_m}{(x-a_m)}, \text{ where } A_1, A_2, \dots, A_m \text{ are constants must be evaluated.}$$

Ex. (1) Find $I = \int \frac{x^2+3x+3}{x^3-x} dx = \int \frac{(x^2+3x+3)}{x(x-1)(x+1)} dx$

$$\begin{aligned} \frac{x^2+3x+3}{x(x-1)(x+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ &= \frac{A(x-1)(x+1) + Bx(x+1) + C(x-1)x}{x(x-1)(x+1)} \end{aligned}$$

$$x^2+3x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$\text{at } x=0 \Rightarrow 3 = A(0-1)(0+1) + 0 + 0 \Rightarrow A = -3$$

$$\text{at } x=1 \Rightarrow 7 = 0 + B(1)(1+1) + 0 \Rightarrow B = \frac{7}{2}$$

$$\text{at } x=-1 \Rightarrow 1 = 0 + 0 + C(-1)(-1-1) \Rightarrow C = \frac{1}{2}$$

OR

$$\begin{aligned} x^2+3x+3 &= Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx \\ &= (A+B+C)x^2 + (B-C)x - A \end{aligned}$$

$$\left. \begin{array}{l} A+B+C = 1 \\ B-C = 3 \\ -A = 3 \end{array} \right\} \Rightarrow A = -3, B = \frac{7}{2}, C = \frac{1}{2}$$

$$I = \int \left[\frac{-3}{x} + \frac{\frac{7}{2}}{x-1} + \frac{\frac{1}{2}}{x+1} \right] dx = -3 \ln|x| + \frac{7}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C$$

Case II Repeated Factors of $Q_m(x)$

(26)

Suppose $(x-a)^r$ is the highest power of $(x-a)$ which divides $Q_m(x)$. Then to this factor we assign the sum of r partial fraction as follows:

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_r}{(x-a)^r}$$

Where A_1, A_2, \dots, A_r are constants must be evaluated.

For example

$$\frac{x^3 - 3x^2 + 4x - 2}{x(x-1)^2(x+1)(x+2)^3} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x+1)} + \frac{E}{(x+2)} + \frac{F}{(x+2)^2} + \frac{G}{(x+2)^3}.$$

Ex.(2) Evaluate $I = \int \frac{(x+5)dx}{(x+2)(x-1)^2}$

$$\frac{x+5}{(x+2)(x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + B(x-1)(x+2) + C(x+2)}{(x+2)(x-1)^2}$$

$$x+5 = A(x-1)^2 + B(x-1)(x+2) + C(x+2)$$

$$\text{at } x=1 \Rightarrow 6 = 0+0+3C \Rightarrow C=2$$

$$\text{at } x=-2 \Rightarrow 3 = 9A+0+0 \Rightarrow A=\frac{1}{3}$$

$$\text{at } x=0 \Rightarrow 5 = \frac{1}{3} - 2B + 0 \Rightarrow B = -\frac{1}{3}$$

$$\therefore I = \int \left[\frac{\frac{1}{3}}{x+2} - \frac{\frac{1}{3}}{x-1} + \frac{2}{(x-1)^2} \right] dx = \frac{1}{3} \ln(x+2) - \frac{1}{3} \ln(x-1) - \frac{2}{(x-1)} + C.$$

Case III Quadratic factors of $Q_m(x)$

Suppose $(x^2+ax+b)^r$ is the highest power of (x^2+ax+b) which divides $Q_m(x)$. Then to this factor we assign the sum of r partial fractions as follows:-

$$\frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \dots + \frac{A_rx+B_r}{(x^2+ax+b)^r}, \text{ where}$$

$A_1, A_2, \dots, A_r, B_1, B_2, \dots, B_r$ are constants must be evaluated
for example

$$\frac{x^2+2x-5}{x^2(x-1)(x^2+1)(x^2+2x+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{x^2+2x+2} + \frac{Hx+I}{(x^2+2x+2)^2}.$$

Ex(B) Evaluate $I = \int \frac{x^5 - x^4 - 3x + 5}{x^4 - 2x^3 + 2x^2 - 2x + 1} dx$

$$\begin{array}{r} x+1 \\ \hline x^4 - 2x^3 + 2x^2 - 2x + 1 \end{array} \overbrace{\begin{array}{r} x^5 - x^4 - 3x + 5 \\ \hline -x^5 + 2x^4 + 2x^3 + 2x^2 - x \\ \hline x^4 - 2x^3 + 2x^2 - 4x + 5 \\ \hline x^4 + 2x^3 + 2x^2 + 2x - 1 \\ \hline -2x + 4 \end{array}}$$

$$I = \int \left(x+1 + \frac{-2x+4}{x^4 - 2x^3 + 2x^2 - 2x + 1} \right) dx = \frac{x^2}{2} + x + \int \frac{(-2x+4) dx}{x^4 - 2x^3 + 2x^2 - 2x + 1}$$

$$\frac{-2x+4}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$-2x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$\text{at } x=1 \Rightarrow 2 = 0 + 2B + 0 \Rightarrow B=1$$

$$\text{at } x=\infty \Rightarrow 4 = A(-1)(1) + B(1) + D(-1)^2 \Rightarrow 4 = -A + B + D \Rightarrow 3 = -A + D$$

$$\text{at } x=-1 \Rightarrow 6 = A(-2)(2) + 2B + (-C+D)(4) \Rightarrow 1 = -A - C + D$$

$$\text{at } x=2 \Rightarrow 0 = A(1)(5) + B(5) + (2C+D)(1) \Rightarrow -5 = 5A + 2(C+D)$$

$$A = -2, B = 1, C = 2, D = 1$$

$$\begin{aligned}
 \text{Let } I_1 &= \int \frac{(-2x+4)dx}{x^4 - 2x^3 + 2x^2 - 2x + 1} = \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1} \right) dx \\
 &= -2 \ln(x-1) - \frac{1}{x-1} + \int \frac{2x dx}{x^2+1} + \int \frac{dx}{x^2+1} \\
 &= -2 \ln(x-1) - \frac{1}{x-1} + \ln(x^2+1) + \tan^{-1} x + C \\
 \therefore I &= \frac{x^2}{2} + x - 2 \ln(x-1) - \frac{1}{x-1} + \ln(x^2+1) + \tan^{-1} x + C.
 \end{aligned}$$

Exercises To Solve [No. 7]

$$(1) \int \frac{x^2+3x+4}{x-2} dx$$

$$(2) \int \frac{x^3-x^2+2x+2}{x^2+3x+2} dx$$

$$(3) \int \frac{x^4+1}{x^3-x} dx$$

$$(4) \int_0^{\sqrt{3}} \frac{5x^2}{x^2+1} dx$$

$$(5) \int \frac{dx}{x^2(x^2+1)^2}$$

$$(6) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$$

$$(7) \int \frac{x^2+3x+3}{(x+1)(x^2+1)} dx$$

$$(8) \int_0^{\ln 2} \frac{e^x dx}{e^{2x} + 3e^x + 2}$$

Method [8] Integration of Irrational Functions

Case I: If the integral contain a single irrational expression of the form $\sqrt[q]{(ax+b)} = (ax+b)^{\frac{1}{q}}$.

$$\text{Let } z = (ax+b)^{\frac{1}{q}} \Rightarrow z^q = ax+b \Rightarrow qz^{q-1} dz = a dx \\ \Rightarrow dx = \frac{q}{a} z^{q-1} dz$$

$$\underline{\text{Ex. (1)}} \quad I = \int \frac{2x+3}{\sqrt{x+2}} dx = \int \frac{2x+3}{(x+2)^{1/2}} dx$$

$$\text{Let } z = (x+2)^{1/2} \Rightarrow z^2 = x+2 \Rightarrow 2z dz = dx$$

$$I = \int \frac{2(z^2-2)+3}{z} \cdot 2z dz = 2 \int (2z^2-1) dz \\ = 2 \left[\frac{2}{3} z^3 - z \right] + C \\ = 2 \left[\frac{2}{3} (x+2)^{3/2} - (x+2)^{1/2} \right] + C.$$

$$\underline{\text{Ex. (2)}} \quad \text{Find } I = \int \frac{\sqrt[3]{x+1}}{x} dx = \int \frac{(x+1)^{1/3}}{x} dx$$

$$\text{Let } z = (x+1)^{1/3} \Rightarrow z^3 = (x+1) \Rightarrow 3z^2 dz = dx$$

$$I = \int \frac{z \cdot 3z^2}{z^3-1} dz = 3 \int \frac{z^3}{z^3-1} dz = 3 \int \left(1 + \frac{1}{z^3-1}\right) dz \\ = 3z + \int \frac{3 dz}{z^3-1}$$

$$\text{let } I_1 = \int \frac{3}{z^3-1} dz = \int \frac{3 dz}{(z-1)(z^2+z+1)}$$

$$\frac{3}{(z-1)(z^2+z+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+z+1}$$

(30)

$$3 = A(z^2 + z + 1) + (Bz + C)(z - 1)$$

$$A=1, \quad B=-1, \quad C=-2$$

$$\begin{aligned} I_1 &= \int \left(\frac{1}{z-1} - \frac{z+2}{z^2+z+1} \right) dz = \ln(z-1) - \frac{1}{2} \int \frac{2z+1-1}{z^2+z+1} dz - \int \frac{z}{z^2+z+1} dz \\ &= \ln(z-1) - \frac{1}{2} \int \frac{(2z+1)^2 + \frac{1}{2}}{z^2+z+1} dz - \int \frac{z}{z^2+z+1} dz \\ &= \ln(z-1) - \frac{1}{2} \ln(z^2+z+1) + \frac{3}{2} \int \frac{dz}{z^2+z+1} \\ &= \ln(z-1) - \frac{1}{2} \ln(z^2+z+1) - \frac{3}{2} \int \frac{dz}{\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}} \end{aligned}$$

$$I_1 = \ln(z-1) - \frac{1}{2} \ln(z^2+z+1) - \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2z+1}{\sqrt{3}/2}\right) + C$$

$$\begin{aligned} I &= 3(x+1)^{1/3} + \ln[(x+1)^{1/3} - 1] - \frac{1}{2} \ln[(x+1)^{2/3} + (x+1)^{1/3} + 1] \\ &\quad - \sqrt{3} \tan^{-1}\left[\frac{2(x+1)^{1/3}}{\sqrt{3}}\right] + C \end{aligned}$$

H.W. EX.(3) Find $\int \frac{dx}{\sqrt[3]{x^2+4\sqrt{x}}}$

EX.(4) Find $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}} dx$

Case II If a single irrational expression of the form

$\sqrt{a^2-x^2}$ or $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$, the z substitution to the radical reduces the given integral to that of a rational function.

EX.(5)

$$\text{Find } I = \int \frac{\sqrt{4-x^2}}{x^3} dx = \int \frac{(4-x^2)^{1/2}}{x^3} dx$$

$$z = (4-x^2)^{1/2} \Rightarrow z^2 = 4-x^2 \Rightarrow x^2 = 4-z^2 \Rightarrow 2x dx = -2z dz$$

$$\Rightarrow x dx = -z dz$$

$$I = \int \frac{(4-x^2)^{1/2}}{x^3} dx = \int \frac{(4-x^2)^{1/2} \cdot x dx}{x^4} = \int \frac{z \cdot (-z) dz}{(4-z^2)^2}$$

$$= - \int \frac{z^2}{(z+2)^2(z-2)} dz$$

$$\frac{-z^2}{(z+2)^2(z-2)^2} = \frac{A}{(z+2)} + \frac{B}{(z+2)^2} + \frac{C}{(z-2)} + \frac{D}{(z-2)^2}$$

$$A = \frac{1}{8}, B = -\frac{1}{4}, C = -\frac{1}{8}, D = -\frac{1}{4}$$

$$I = \int \left[\frac{\frac{1}{8}}{z+2} - \frac{\frac{1}{4}}{(z+2)^2} - \frac{\frac{1}{8}}{z-2} - \frac{\frac{1}{4}}{(z-2)^2} \right] dz$$

$$= \frac{1}{8} \ln(z+2) + \frac{1}{4} \frac{1}{z+2} - \frac{1}{8} \ln(z-2) + \frac{1}{4} \frac{1}{z-2} + C$$

$$= \frac{1}{8} \ln\left(\frac{z+2}{z-2}\right) + \frac{1}{4} \left[\frac{1}{z+2} + \frac{1}{z-2} \right] + C$$

$$= \frac{1}{8} \ln\left(\frac{z+2}{z-2}\right) + \frac{1}{4} \left[\frac{2z}{z^2-4} \right] + C$$

$$= \frac{1}{8} \ln\left(\frac{z+2}{z-2}\right) - \frac{1}{2} \left[\frac{z}{4-z^2} \right] + C$$

$$= \frac{1}{8} \ln\left(\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right) - \frac{1}{2} \left[\frac{\sqrt{4-x^2}}{4-x^2} \right] + C$$

Exercises To Solve [No. 8]

$$\textcircled{1} \int \frac{\sqrt{x+2}}{\sqrt{x}-1} dx$$

$$\textcircled{2} \int \frac{2x+1}{(x+2)^{2/3}} dx$$

$$\textcircled{3} \int \frac{dx}{\sqrt{x+3}\sqrt[3]{x}}$$

$$\textcircled{4} \int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}}$$

$$\textcircled{5} \int \frac{\sqrt{x^2+9}}{x^3} dx$$

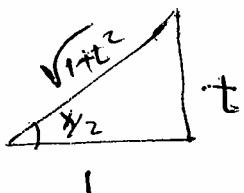
Method [9] Integration of Rational Functions of Trigonometric

If the integrand is a rational function of trigonometric, the substitution of $t = \tan \frac{x}{2}$ will reduce the integral to a rational function of t which can be handle by method [7].

Mathematically Speaking :-

$$t = \tan \frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1} t \Rightarrow \frac{1}{2} dx = \frac{dt}{1+t^2}$$

$$\Rightarrow dx = \frac{2dt}{1+t^2}$$



$$\text{Since } \sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}}, \cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1-t^2}{1+t^2}$$

Ex.(1) $I = \int \frac{dx}{\cancel{5-4\cos x}} = \int \frac{\frac{2dt}{1+t^2}}{5-4 \frac{1-t^2}{1+t^2}} = \int \frac{2dt}{5(1+t^2) - 4(1-t^2)}$

 $= 2 \int \frac{dt}{1+9t^2} = \frac{2}{3} \int \frac{3dt}{1+(3t)^2} = \frac{2}{3} \tan^{-1} 3t + C$
 $= \frac{2}{3} \tan^{-1} \left[3 \left(\tan \frac{x}{2} \right) \right] + C.$

H.W
Ex.(2) $I = \int \frac{dx}{3\cos x + 4\sin x}$

Exercises To solve [No. 9]

(1) $\int \frac{dx}{2-\sin x}$

(2) $\int \frac{\cos x dx}{5+4\cos x}$

(3) $\int_0^{\pi} \frac{dx}{1+\sin x}$

(4) $\int_{\frac{\pi}{2}}^{\pi} \frac{dx}{1-\cos x}$

(5) $\int \frac{dx}{2-\cos x}$

(6) $\int \frac{dx}{5+4\cos x}$